

PARAMETRIZATIONS OF DEGENERATE DENSITY MATRICES

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ABSTRACT. It turns out that a parametrization of degenerate density matrices requires a parametrization of $\mathfrak{F} = U(n)/(U(k_1) \times U(k_2) \times \cdots \times U(k_m))$ $n = k_1 + \cdots + k_m$ where $U(k)$ denotes the set of all unitary $k \times k$ -matrices with complex entries. Unfortunately the parametrization of this quotient space is quite involved. Our solution does not rely on Lie algebra methods directly, but succeeds through the construction of suitable sections for natural projections, by using techniques from the theory of homogeneous spaces. We mention the relation to the Lie algebra background and conclude with two concrete examples.

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1. INTRODUCTION

In various parts of Physics density matrices, i.e., positive trace class operators of trace 1 on a complex separable Hilbert space play an important role, see [5]. Density matrices represent states of quantum systems. In many concrete applications the Hilbert space is typically finite dimensional and the Hilbert space then is \mathbb{C}^n , the space of n -tuples of complex numbers with its standard inner product. Thus the space \mathcal{D}_n of all density matrices on \mathbb{C}^n is the space of all $n \times n$ matrices ρ with complex entries such that $\langle x, \rho x \rangle \geq 0$ for all $x \in \mathbb{C}^n$ and $\text{Tr}(\rho) = \sum_{j=1}^n \langle e_j, \rho e_j \rangle = 1$ for any orthonormal basis $\{e_j : j = 1, \dots, n\}$ of \mathbb{C}^n . Through these two constraints the entries of a density matrix are not all independent and thus contain redundant parts. But for an effective description of quantum states one would like to get rid of these redundant parts of a density matrix, i.e., one would like to have a description of density matrices in terms of a set of independent parameters, that is a parametrization in the sense of Definition 1.1. The best known parametrization of density matrices seems to be the Bloch vector parametrization [4, 14]. While this parametrization is perfect for $n = 2$ -level systems, it has a serious defect for $n \geq 3$ -level systems in the sense that the parameter set cannot be determined explicitly (see for instance [9]). Thus various authors have been looking for alternative ways to parametrize density matrices, see for instance [2, 8, 10, 15]. Some time ago we started with a parametrization of density matrices based on their spectral representation [7, 6, 9].

The spectral representation of a density matrix $\rho \in \mathcal{D}_n$ reads

$$(1.1) \quad \rho = U D_n(\lambda_1, \dots, \lambda_n) U^*$$

where $D_n(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix of the eigenvalues $\lambda_1, \dots, \lambda_n$ and U is some unitary $n \times n$ matrix, i.e., $U \in U(n)$. These n eigenvalues are not necessarily distinct; they occur in this list as many times as their multiplicity requires.

In this article we consider parametrizations in the strict sense as suggested in [9]. This definition reads:

Definition 1.1. A parametrization of density matrices is given by the following:

- (a) Specification of a parameter set $Q_n \subset \mathbb{R}^m$ where m depends on n , i.e., $m = m(n)$;
- (b) Specification of a one-to-one and onto map $F_n : Q_n \longrightarrow \mathcal{D}_n$.

When the spectral representation (1.1) is chosen as the starting point one obviously needs a suitable parametrization of unitary matrices.

The set of eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ can be ordered according to their size: We denote the set of eigenvalues ordered in this way by Λ_n , i.e.,

$$(1.2) \quad \Lambda_n = \left\{ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) : 0 \leq \lambda_n \leq \dots \leq \lambda_2 \leq \lambda_1, \sum_{j=1}^n \lambda_j = 1 \right\}.$$

We begin by addressing the question of uniqueness of the spectral representation (1.1). Accordingly suppose that for $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Lambda_n$ and $U, V \in U(n)$ we have

$$U^* D_n(\boldsymbol{\lambda}) U = V^* D_n(\boldsymbol{\lambda}') V$$

Since the spectrum of a matrix is uniquely determined and since $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Lambda_n$ it follows $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ and therefore it follows $VU^* D_n(\boldsymbol{\lambda}) = D_n(\boldsymbol{\lambda}') VU^*$, i.e.,

$$(1.3) \quad VU^* \in D_n(\boldsymbol{\lambda})'$$

where $D_n(\boldsymbol{\lambda})'$ denotes the commutant of the diagonal matrix $D_n(\boldsymbol{\lambda})$ in $U(n)$. If a density matrix $\rho \in \mathcal{D}_n$ has a non-degenerate spectrum, i.e., if

$$(1.4) \quad \boldsymbol{\lambda} \in \Lambda_n^\# = \{\boldsymbol{\lambda} \in \Lambda_n : 0 \leq \lambda_n < \lambda_{n-1} < \dots < \lambda_2 < \lambda_1\}$$

then this commutant is easily determined and is given by

$$(1.5) \quad \mathcal{D}_n(\boldsymbol{\lambda})' = U(1) \times \dots \times U(1), \quad n \text{ terms}$$

Naturally there are many ways in which a density matrix can be degenerate. Suppose that the spectrum of $\rho_n \in \mathcal{D}_n$ has m different eigen-values $\lambda_1, \dots, \lambda_m$ with multiplicities k_1, \dots, k_m with $\sum_{j=1}^m k_j = n$. Thus $\boldsymbol{\lambda} \in \Lambda_n$ is of the form

$$(\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_m, \dots, \lambda_m)$$

where each λ_j is repeated k_j times and $\sum_{j=1}^m k_j \lambda_j = 1$ and where we use the ordering $0 \leq \lambda_m < \lambda_{m-1} < \dots < \lambda_1$ according to (1.2). Thus one has in this case

$$(1.6) \quad D_n(\boldsymbol{\lambda}) = \text{diag}_n(\lambda_1 I_{k_1}, \lambda_2 I_{k_2}, \dots, \lambda_m I_{k_m})$$

where diag_n denotes the $n \times n$ diagonal matrix with entries as indicated and where I_{k_j} denotes the $k_j \times k_j$ identity matrix. Therefore the commutant of the diagonal matrix $D_n(\boldsymbol{\lambda})$ is in this case

$$(1.7) \quad D_n(\boldsymbol{\lambda})' = U(k_1) \times U(k_2) \times \dots \times U(k_m).$$

Thus in order to complete the parametrization problem for degenerate density matrices we need to find a suitable parametrization of

$$(1.8) \quad \mathfrak{F} = U(n)/(U(k_1) \times U(k_2) \times \dots \times U(k_m)) \quad n = k_1 + \dots + k_m.$$

We begin with a discussion of the simplest case, i.e., $k_j = 1$ for all j and $m = n$. Note that in this case

$$U(n)/(U(1) \times \dots \times U(1)) = U(n)/ \sim$$

for the equivalence relation \sim in $U(n)$ defined by

$$U \sim U' \Leftrightarrow U^{-1}U' \in U(1) \times \dots \times U(1).$$

Accordingly the elements of $U(n)/\sim$ are the equivalence classes

$$[U] = \{UV; V \in U(1) \times \dots \times U(1)\}$$

$U(n)/\sim = U(n)/(U(1) \times \dots \times U(1))$ is called a (complex) full flag manifold (see [13]). Introduce the natural projection

$$\pi : U(n) \ni U \rightarrow [U] \in U(n)/(U(1) \times \dots \times U(1)).$$

Then a map $\iota : U(n)/(U(1) \times \dots \times U(1)) \rightarrow U(n)$ is called a **section of $U(n)$ on $U(n)/(U(1) \times \dots \times U(1))$** for π if $\pi \circ \iota = \text{id}$. The relation $U' = \iota([U])$ implies that U' is a representative of the coset $[U]$. The mapping

$$p : \Lambda_n^\neq \times U(n)/(U(1) \times \dots \times U(1)) \ni ((\lambda_1, \dots, \lambda_n), m) \longrightarrow \iota(m)D_n(\lambda_1, \dots, \lambda_n)\iota(m)^*$$

does not depend on the section ι , and thus the mapping p gives a parametrization of density matrices, if $U(n)/(U(1) \times \dots \times U(1))$ is suitably parametrized. Since $U(n)/(U(1) \times \dots \times U(1))$ is a manifold, it is parametrized locally. But unfortunately, this parametrization is not simple. Even though the mapping does not depend on ι , the construction of a concrete section is necessary, but also not so simple.

Through the construction of a concrete section we will also achieve a parametrization of unitary matrices, an important problem in itself which has found considerable attention in the last 10 – 15 years (see the references mentioned above). The starting point of this construction is the so called canonical coset decomposition which gives in particular the well-known Jarlskog parametrization [11, 12].

Recall that the coset space $U(n)/(U(n-1) \times U(1))$ is the projective space CP^{n-1} (see [13]).

Symbolically, the canonical coset decomposition is:

$$\begin{aligned} U(n) &= U(n)/(U(n-1) \times U(1)) \cdot U(n-1)/(U(n-2) \times U(1)) \\ &\quad \cdot \dots \cdot U(2)/(U(1) \times U(1)) \cdot (U(1) \times \dots \times U(1)). \end{aligned}$$

In Section 3, we parametrize $U(n)$ by constructing sections $\iota_j : CP^{n-j} \rightarrow U(n-j+1)$ for $j = 1, \dots, n-1$.

For the degenerate case we have only to use (complex) Grassmann manifolds $U(k)/(U(k_1) \times U(k_2))$ instead of the projective spaces $U(k)/(U(k-1) \times U(1))$:

$$\begin{aligned} U(n) &= U(n)/(U(n-k_m) \times U(k_m)) \cdot U(n-k_m)/(U(n-k_{m-1}-k_m) \times U(k_{m-1})) \\ &\quad \cdot \dots \cdot U(k_1+k_2)/(U(k_1) \times U(k_2)) \cdot (U(k_1) \times \dots \times U(k_r)). \end{aligned}$$

In Section 2, we study this case extensively because the parametrization of degenerate density matrices is the main new result of this paper (Propositons 2.10, 2.13). The result of Section 3 is the special case of $k_j = 1$ for $1 \leq j \leq n$. In Section 4, we study Jarlskog parametrization used in [9] for the non-degenerate case.

If $S(\mathbb{C}^n)$ denotes the unit sphere in \mathbb{C}^n , we can parametrize the subset $\Omega = \{[z]; z \in S(\mathbb{C}^n), z_n \neq 0\} \subset CP^{n-1}$ by $B(n-1) = \{x \in \mathbb{C}^{n-1}; \|x\| < 1\}$. But for the boundary $\partial B(n-1) = \{x \in \mathbb{C}^{n-1}; \|x\| = 1\}$, the mapping

$$\partial B(n-1) \ni x \rightarrow [x] \in CP^{n-1}$$

is not injective and consequently there are x and x' in $\partial B(n-1)$ such that $W(x) \neq W(x')$ for $W(x)$ of (3.5) and there exist V and V' in $U(n-1) \times U(1)$ such that

$$W(x)V = W(x')V'.$$

Consequently, the parametrization of density matrices is not always unique. In Section 5 we present a way to construct a section on a Grassmann manifold by using sections on a suitable projective space, since, for concrete calculations, the construction of a section presented in Section 2 is fairly involved. In Section 7, we give simple concrete examples of degenerate

density matrices. In this paper, we mainly use the technique of homogeneous spaces. But there is the theory of Lie algebra behind it. In Section 6, a Lie algebraic back ground is presented.

2. GRASSMANNIAN AND CANONICAL COSET DECOMPOSITION

The Grassmann manifold $G(k, \mathbb{C}^n)$ is the set of all complex k -dimensional subspaces of \mathbb{C}^n (see [13]). Let W be a k -dimensional subspace of \mathbb{C}^n . Then we choose a basis of column vectors z_1, \dots, z_k of W and associate with it the matrix

$$M(W) = (z_1 \ z_2 \ \dots \ z_k).$$

Since the matrix $M(W)$ depends on the choice of a basis of W , $M(W)$ is not determined uniquely by W . There is a freedom of multiplication by regular $k \times k$ matrices from the right. Thus we have

$$G(k, \mathbb{C}^n) = \{M; M = \text{complex } n \times k \text{ matrix of rank } k\} / GL(k, \mathbb{C}).$$

Let $S_{k,n}$ be the set of permutations σ of $\{1, \dots, n\}$ such that $1 \leq \sigma(1) < \sigma(2) < \dots < \sigma(n-k) \leq n$ and $1 \leq \sigma(n-k+1) < \sigma(n-k+2) < \dots < \sigma(n) \leq n$. Let $M(W)_{\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)}$ be the $k \times k$ -matrix which consists of the $\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)$ -th rows of $M(W)$, and define

$$\Omega_\sigma = \{W \in G(k, \mathbb{C}^n); \det M(W)_{\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)} \neq 0\} \subset G(k, \mathbb{C}^n).$$

Since the rank of the matrix $M(W)$ is k , there is a $\sigma \in S_{k,n}$ such that $\det M(W)_{\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)} \neq 0$. Thus we have

$$(2.1) \quad G(k, \mathbb{C}^n) = \cup_{\sigma \in S} \Omega_\sigma.$$

Let $M(n-k, k)$ be a set of all $(n-k) \times k$ complex matrices. Define the mapping

$$(2.2) \quad \phi_\sigma : \Omega_\sigma \ni W \rightarrow M(W)_{\sigma(1), \dots, \sigma(n-k)} M(W)_{\sigma(n-k+1), \dots, \sigma(n)}^{-1} \in M(n-k, k).$$

Then ϕ_σ gives the homeomorphism

$$\Omega_\sigma \cong M(n-k, k) \cong \mathbb{C}^{(n-k)k}.$$

In fact, the element W of Ω_σ corresponds in a 1-1 way to the matrix of the form

$$(2.3) \quad M(W)_\sigma M(W)_{\sigma(n-k+1), \dots, \sigma(n)}^{-1} = \begin{pmatrix} \phi_\sigma(W) \\ I \end{pmatrix} = \begin{pmatrix} Z \\ I \end{pmatrix} = \begin{pmatrix} z_{11} & \dots & z_{1k} \\ \vdots & \ddots & \vdots \\ z_{n-k,1} & \dots & z_{n-k,k} \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix},$$

where $M(W)_\sigma$ is the matrix whose i -th row is the $\sigma(i)$ -th row of $M(W)$, $Z \in M(n-k, k)$ and I is the $k \times k$ identity matrix. Then the set $\{(\Omega_\sigma, \phi_\sigma); \sigma \in S_{k,n}\}$ gives an atlas of $G(k, \mathbb{C}^n)$.

There is another parametrization of Ω_σ which is more convenient for our purpose.

Let

$$B(n-k, k) = \{X \in M(n-k, k); X^* X < I_k\}.$$

Then the set Ω_σ can be parametrized by the set $B(n-k, k)$. We will show this in the following.

Let $S(k, \mathbb{C}^n)$ denote the set of all orthonormal frames $F = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k)$ in \mathbb{C}^n of length k , i.e., $\langle \mathbf{x}_j, \mathbf{x}_i \rangle = \delta_{ij}$ for $i, j = 1, \dots, k$. For $F \in S(k, \mathbb{C}^n)$ we associate a matrix $M(F)$ by

$$M(F) = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k).$$

Let $M(F)_{\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)}$ ($F \in S(k, \mathbb{C}^n)$) be the $k \times k$ -matrix which consists of $\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)$ -th rows of $M(F)$, and let

$$(2.4) \quad \tilde{\Omega}_\sigma = \{F \in S(k, \mathbb{C}^n); \det M(F)_{\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)} \neq 0\} \subset S(k, \mathbb{C}^n).$$

Since the rank of the matrix $M(F)$ is k , there is a $\sigma \in S_{k,n}$ such that $\det M(F)_{\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)} \neq 0$. Thus we have

$$S(k, \mathbb{C}^n) = \cup_{\sigma \in S_{k,n}} \tilde{\Omega}_\sigma.$$

Definition 2.1. For a permutation σ of $\{1, 2, \dots, n\}$ define $U_\sigma \in U(n)$ by $U_\sigma e_j = e_{\sigma(j)}$.

Then $M(F)_\sigma = U_\sigma^{-1} M(F)$. Define F_σ by $M(F_\sigma) = M(F)_\sigma$, and identify F and $M(F)$. Let π_2 be the surjective mapping

$$(2.5) \quad \pi_2 : S(k, \mathbb{C}^n) \ni F \rightarrow W = \text{span } F \in G(k, \mathbb{C}^n),$$

where $\text{span } F$ is the complex subspace of \mathbb{C}^n spanned by the frame F . If $F, F' \in S(k, \mathbb{C}^n)$ define the same subspace, then $F' = FU$ for some $U \in U(k)$. Thus we have

$$(2.6) \quad G(k, \mathbb{C}^n) \cong S(n, \mathbb{C}^n)/U(k).$$

In order to parametrize $G(k, \mathbb{C}^n)$, we must choose a unique representative $F \in S(k, \mathbb{C}^n)$ from (2.6). Note that

$$(2.7) \quad G(k, \mathbb{C}^n) \supset \Omega_\sigma = \{W \in G(k, \mathbb{C}^n); \det F(W)_{\sigma(n-k+1), \sigma(n-k+2), \dots, \sigma(n)} \neq 0\} = \pi_2(\tilde{\Omega}_\sigma).$$

For $W \in \Omega_\sigma$, we can choose a unique representative from the coset $F(W)U(k)$. In fact, since the submatrix $Y_\sigma = F_{\sigma(n-k+1), \dots, \sigma(n)}$ is nonsingular, from the uniqueness of the polar decomposition (see [3]) we have

$$(2.8) \quad Y_\sigma^* = U|Y_\sigma^*|$$

for a unique $U \in U(k)$. Consequently

$$Y_\sigma = |Y_\sigma^*|U^*$$

for a unique $U^* \in U(k)$. So, we select a unique representative $\begin{pmatrix} X'_\sigma \\ Y'_\sigma \end{pmatrix}$ which corresponds to $W \in \Omega_\sigma$ such that $Y'_\sigma = Y_\sigma U = |Y_\sigma^*|$ is a positive operator and $X'_\sigma = X_\sigma U = F_{\sigma(1), \dots, \sigma(n-k+1)} U$. Since the column vectors of $\begin{pmatrix} X'_\sigma \\ Y'_\sigma \end{pmatrix}$ gives an orthonormal frame, we have

$$X'^*_\sigma X'_\sigma + Y'^2_\sigma = (X'^*_\sigma, Y'_\sigma) \begin{pmatrix} X'_\sigma \\ Y'_\sigma \end{pmatrix} = I_k.$$

This shows that $X'_\sigma \in B(n-k, k)$ and $Y'_\sigma = (I_k - X'^*_\sigma X'_\sigma)^{1/2}$. Thus Ω_σ is parametrized by $B(n-k, k)$.

This can be also understood (by showing directly that there is a 1 to 1 correspondence between $B(n-k, k)$ and $M(n-k, k)$. For the matrix Z of (2.3) introduce

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} Z \\ I \end{pmatrix} (Z^* Z + I)^{-1/2}$$

Then we have

$$I - X^* X = Y^2 > 0, \text{ and } 0 \leq X^* X < I \text{ and } Y = (I - X^* X)^{1/2}.$$

Therefor the mappings

$$\begin{pmatrix} Z \\ I \end{pmatrix} \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix}$$

and

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} (I - X^* X)^{-1/2} = \begin{pmatrix} Z \\ I \end{pmatrix}$$

give a 1-1 onto correspondence, and the mappings

$$M(n-k, k) \ni Z \rightarrow X = Z(Z^* Z + I)^{-1/2} \in B(n-k, k),$$

$$B(n-k, k) \ni X \rightarrow Z = X(I - X^* X)^{-1/2} \in M(n-k, k)$$

give a 1-1 onto correspondence between $M(n-k, k)$ and $B(n-k, k)$.

Let $\tilde{\psi}_\sigma$ be the mapping

$$\tilde{\psi}_\sigma : \tilde{\Omega}_\sigma \ni F \rightarrow X_\sigma U \in B(n-k, k), \quad F_\sigma = \begin{pmatrix} X_\sigma \\ Y_\sigma \end{pmatrix}, \quad Y_\sigma U = |Y_\sigma^*|.$$

Then $\tilde{\psi}_\sigma$ induces the mapping

$$\psi_\sigma : \Omega_\sigma \ni \pi_2(F) \rightarrow X_\sigma U \in B(n-k, k), \quad F_\sigma = \begin{pmatrix} X_\sigma \\ Y_\sigma \end{pmatrix}, \quad Y_\sigma U = |Y_\sigma^*|$$

because $\pi_2(F) = \pi_2(F')$ implies $\psi_\sigma(F) = \psi_\sigma(F')$.

Proposition 2.2. *The mapping $\kappa_\sigma : B(n-k, k) \rightarrow \Omega_\sigma = \pi_2(\tilde{\Omega}_\sigma)$ defined by*

$$\kappa_\sigma : B(n-k, k) \ni X \rightarrow U_\sigma \pi_2 \left(\begin{pmatrix} X \\ (I_k - X^* X)^{1/2} \end{pmatrix} \right) \in \Omega_\sigma$$

satisfies $\kappa_\sigma \circ \psi_\sigma = \text{id}$ and $\psi_\sigma \circ \kappa_\sigma = \text{id}$.

Proof. This is obvious. □

It is easily seen that for any $F, F' \in S(k, \mathbb{C}^n)$ there exists $U \in U(n)$ such that $F = UF'$, that is $U(n)$ acts transitively on $S(k, \mathbb{C}^n)$. Let $x = (e_{n-k+1}, e_{n-k+2}, \dots, e_n) \in S(k, \mathbb{C}^n)$. Then the isotropy subgroup of $U(n)$ at x is $U(n-k) \times \{I_k\}$.

Let $F, F' \in S(k, \mathbb{C}^n)$, and suppose $\pi_2(F) = \pi_2(F')$. Then there exists $Q \in U(k)$ such that $F = F'Q$. Let $U \in U(n)$. Then $UF = UF'Q$, i.e., $\pi_2(UF) = \pi_2(UF')$. This shows that $U(n)$ acts on $G(k, \mathbb{C}^n) = S(k, \mathbb{C}^n)/U(k)$ by

$$U\pi_2(F) = \pi_2(UF).$$

Since $U(n)$ acts on $S(k, \mathbb{C}^n)$ transitively, $U(n)$ acts on $G(k, \mathbb{C}^n) = \pi_2(S(k, \mathbb{C}^n))$ transitively. This shows that $G(k, \mathbb{C}^n)$ is a homogeneous space of $U(n)$ (see [13]). Let $y \in G(k, \mathbb{C}^n)$ be a k dimensional subspace of \mathbb{C}^n spanned by the vectors $(e_{n-k+1}, e_{n-k+2}, \dots, e_n)$. The isotropy subgroup of $U(n)$ at y is $U(n-k) \times U(k)$ and thus we have

$$U(n)/(U(n-k) \times U(k)) \cong G(k, \mathbb{C}^n).$$

For

$$x = (e_{n-k+1}, e_{n-k+2}, \dots, e_n) = \begin{pmatrix} O \\ I_k \end{pmatrix} \in S(k, \mathbb{C}^n)$$

and

$$g = \begin{pmatrix} W & X \\ V & Y \end{pmatrix} \in U(n),$$

denote by $F = \begin{pmatrix} X \\ Y \end{pmatrix}$ the last k columns of g with a $k \times k$ matrix Y . Now define the mapping $\pi_1 : U(n) \rightarrow S(k, \mathbb{C}^n)$ by

$$(2.9) \quad \pi_1(g) = gx = \begin{pmatrix} W & X \\ V & Y \end{pmatrix} \begin{pmatrix} O \\ I_k \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \in S(k, \mathbb{C}^n).$$

Suppose that Y is a regular $k \times k$ matrix, i.e., $F \in \tilde{\Omega}_e (= \tilde{\Omega}_\sigma$ for $\sigma = e$ the identity permutation). Then there is a unique $Q \in U(k)$ such that $YQ = |Y^*|$. Put

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} XQ \\ |Y^*| \end{pmatrix}.$$

Then we have

$$\pi_2 \begin{pmatrix} X \\ Y \end{pmatrix} = \pi_2 \begin{pmatrix} X' \\ Y' \end{pmatrix}.$$

Let

$$(2.10) \quad g' = W(X') = \begin{pmatrix} (I - X'X'^*)^{1/2} & X' \\ -X'^* & Y' \end{pmatrix}.$$

Then $g' \in U(n)$ by Proposition 6.6 and we have

$$\pi_1(g') = \begin{pmatrix} (I - X'X'^*)^{1/2} & X' \\ -X'^* & Y' \end{pmatrix} x = \begin{pmatrix} X' \\ Y' \end{pmatrix}.$$

For

$$(2.11) \quad \pi = \pi_2 \circ \pi_1.$$

we have

$$\pi(g) = \pi(g').$$

Definition 2.3. A mapping $\iota : G(k, \mathbb{C}^n) \rightarrow U(n)$ is a section of $U(n)$ on $G(k, \mathbb{C}^n)$ for $\pi : U(n) \rightarrow G(k, \mathbb{C}^n)$ if it satisfies

$$(2.12) \quad \pi(\iota(x)) = x \quad \text{for all } x \in G(k, \mathbb{C}^n).$$

If ι is defined only on a subset $\Omega \subset G(k, \mathbb{C}^n)$ and satisfies (2.12) there, ι is called a local section.

Thus, by (2.10), we have constructed a local section:

Proposition 2.4. *Let*

$$(2.13) \quad W(X) = \begin{pmatrix} (I_{n-k} - XX^*)^{1/2} & X \\ -X^* & (I_k - X^*X)^{1/2} \end{pmatrix}$$

for $X \in B(n-k, k)$. Then the mapping

$$(2.14) \quad \iota_e = W \circ \psi_e : \Omega_e \ni \pi(g) \rightarrow g' \in U(n).$$

gives a local section of $U(n)$ on Ω_e for $\pi : U(n) \rightarrow G(k, \mathbb{C}^n)$.

Proof. In Proposition 6.6 it is shown in detail that the matrix $W(X)$ in (2.13) is unitary. The proof of the remaining part of the statement is straight forward. \square

From now on, we use the notation $\tilde{\Omega}$ for $\tilde{\Omega}_e$, and Ω for Ω_e , where e is the identity permutation.

Proposition 2.5.

$$(2.15) \quad \tilde{\Omega}_\sigma = U_\sigma \tilde{\Omega}, \text{ and } \Omega_\sigma = U_\sigma \Omega.$$

Proof. Let $F \in \tilde{\Omega}_\sigma$. Then $U_\sigma^{-1}F = F_\sigma = \begin{pmatrix} X_\sigma \\ Y_\sigma \end{pmatrix}$ belongs to $\tilde{\Omega}$ since Y_σ is regular. Let $F \in \tilde{\Omega}$ and $F' = U_\sigma F$. Then $F' \in \tilde{\Omega}_\sigma$. In fact, $F'_\sigma = U_\sigma^{-1}F' = F$ and $Y'_\sigma = Y$ is regular. Thus we have $\tilde{\Omega}_\sigma = U_\sigma \tilde{\Omega}$, and by (2.7)

$$\Omega_\sigma = \pi_2(\tilde{\Omega}_\sigma) = \pi_2(U_\sigma \tilde{\Omega}) = U_\sigma \pi_2(\tilde{\Omega}) = U_\sigma \Omega.$$

\square

In Section 6 it is explained why we consider $W(X)$ of (2.13). Now we extend Proposition 2.4 to Ω_σ .

Proposition 2.6. *The mapping*

$$W_\sigma = U_\sigma W : B(n-k, k) \ni X \rightarrow U_\sigma W(X)$$

satisfies $\psi_\sigma \circ \pi \circ W_\sigma(X) = X$.

Proof. For $X \in B(n-k, k)$ we find

$$\pi_1 \circ W_\sigma(X) = \pi_1(U_\sigma W(X)) = U_\sigma \begin{pmatrix} X \\ (I_k - X^*X)^{1/2} \end{pmatrix} = U_\sigma F = F',$$

and $\psi_\sigma \circ \pi_2(F') = \psi_e U_\sigma^{-1}\{F'U; U \in U(k)\} = \psi_e\{FU; U \in U(k)\} = X$. Thus we have $\psi_\sigma \circ \pi \circ W_\sigma(X) = \psi_\sigma \circ \pi_2 \circ \pi_1 \circ W_\sigma(X) = X$. \square

Corollary 2.7. *The mapping*

$$\iota_\sigma = U_\sigma W \circ \psi_\sigma : \Omega_\sigma \ni x \rightarrow W(\psi_\sigma(x)) \in U(n)$$

gives a local section of $U(n)$ on $\Omega_\sigma \subset G(k, \mathbb{C}^n)$ for $\pi : U(n) \rightarrow G(k, \mathbb{C}^n)$.

Proof. Proposition 2.2 shows that ψ_σ is bijective; by Proposition 2.6 we conclude. \square

Let S_n be the set of all permutations of $\{1, 2, \dots, n\}$, and define an order on S_n as follows (lexicographic ordering). Let $\sigma, \sigma' \in S_n$ then $\sigma < \sigma'$ if there exists $s \in \{1, 2, \dots, n\}$ such that $\sigma(j) = \sigma'(j)$ ($j = 1, \dots, s-1$) and $\sigma(s) < \sigma'(s)$. Then $S_{k,n} \subset S_n$ is a well-ordered set, and $S_{k,n} = \{\sigma_1 < \dots < \sigma_m\}$ for $m = \binom{n}{k}$.

For

$$(2.16) \quad V_j = \Omega_{\sigma_j} \setminus \bigcup_{i=1}^{j-1} \Omega_{\sigma_i}$$

one finds that

$$G(k, \mathbb{C}^n) = \bigcup_{j=1}^m V_j, \quad m = \binom{n}{k}$$

is a disjoint union. We can construct a section $\iota : G(k, \mathbb{C}^n) \rightarrow U(n)$ for $\pi : U(n) \rightarrow G(k, \mathbb{C}^n)$ as follows.

Definition 2.8. Let $x \in G(k, \mathbb{C}^n)$. Define the section $\iota(x)$ by

$$\iota(x) = \iota_{\sigma_j}(x) \text{ if } x \in V_j.$$

Proposition 2.9. *Let ι be a section of $U(n)$ on $G(k, \mathbb{C}^n)$ for π . Then for any $g \in U(n)$, there is a unique h such that*

$$g = \iota(\pi(g))h, \quad h \in U(n-k) \times U(k).$$

Proof. Let $g' = \iota(\pi(g))$. Since $\pi(g) = \pi(g')$ and

$$\pi(g) = \{gh; h \in U(n-k) \times U(k)\},$$

there exists $h \in U(n-k) \times U(k)$ such that $g = g'h$. If g has two expression $g = g'h = g'h'$. Then we have $h = g'^{-1}g = h'$. \square

Now we come to the parametrization of unitary matrices by the canonical coset decomposition. Symbolically the canonical coset decomposition is:

$$\begin{aligned} U(n) &= U(n)/(U(n-k_m) \times U(k_m)) \cdot U(n-k_m)/(U(n-k_{m-1}-k_m) \times U(k_{m-1})) \\ &\quad \cdots \cdots U(k_1+k_2)/(U(k_1) \times U(k_2)) \cdot (U(k_1) \times \cdots \times U(k_m)). \end{aligned}$$

The following Proposition shows the precise meaning of the above formula.

Proposition 2.10. *For any section ι_j of $U(n-k_{j+1}-\cdots-k_m)$ on $G(k_j, \mathbb{C}^{n-k_{j+1}-\cdots-k_m})$ for $\pi_j : U(n-k_{j+1}-\cdots-k_m) \rightarrow G(k_j, \mathbb{C}^{n-k_{j+1}-\cdots-k_m})$ there is a unique surjection*

$$\begin{aligned} f : U(n) \ni g \rightarrow (z_m, z_{m-1}, \dots, z_2) &\in G(k_m, \mathbb{C}^n) \times G(k_{m-1}, \mathbb{C}^{n-k_m}) \times \\ &\cdots \times G(k_2, \mathbb{C}^{n-k_3-\cdots-k_m}) \end{aligned}$$

and a unique $h \in U(k_1) \times \cdots \times U(k_m)$ such that

$$g = \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) & 0 \\ 0 & I_{k_m} \end{pmatrix} \cdots \begin{pmatrix} \iota_2(z_2) & 0 \\ 0 & I_{k_3+\cdots+k_m} \end{pmatrix} h.$$

Proof. According to Proposition 2.9 there exists a unique $H_m = (g_m, h_m) \in U(n-k_m) \times U(k_m)$ such that

$$g = \iota_m(z_m)H_m = \iota_m(z_m)(g_m, h_m) = \iota_m(z_m) \begin{pmatrix} g_m & 0 \\ 0 & h_m \end{pmatrix},$$

where $z_m = \pi_m(g)$.

In the same way, we have a unique element $H_{m-1} = (g_{m-1}, h_{m-1}) \in U(n - k_m - k_{m-1}) \times U(k_{m-1})$ such that

$$g_m = \iota_{m-1}(z_{m-1})H_{m-1} = \iota_{m-1}(z_{m-1})(g_{m-1}, h_{m-1}) = \iota_{m-1}(z_{m-1}) \begin{pmatrix} g_{m-1} & 0 \\ 0 & h_{m-1} \end{pmatrix},$$

for $z_{m-1} = \pi_{m-1}(g_m)$, and

$$\begin{aligned} g &= \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) \begin{pmatrix} g_{m-1} & 0 \\ 0 & h_{m-1} \end{pmatrix} & 0 \\ 0 & h_m \end{pmatrix} \\ &= \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) & 0 \\ 0 & I_{k_m} \end{pmatrix} \begin{pmatrix} g_{m-1} & 0 & 0 \\ 0 & h_{m-1} & 0 \\ 0 & 0 & h_m \end{pmatrix}. \end{aligned}$$

Continuing this procedure, we arrive at

$$g = \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) & 0 \\ 0 & I_{k_m} \end{pmatrix} \cdots \begin{pmatrix} \iota_2(z_2) & 0 \\ 0 & I_{k_3+\dots+k_m} \end{pmatrix} \begin{pmatrix} h_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & h_m \end{pmatrix}.$$

The relation

$$f(g') = (z_m, z_{m-1}, \dots, z_2)$$

for

$$(z_m, z_{m-1}, \dots, z_2) \in G(k_m, \mathbb{C}^n) \times G(k_{m-1}, \mathbb{C}^{n-k_m}) \times \dots \times G(k_2, \mathbb{C}^{n-k_3-\dots-k_m})$$

and

$$g' = \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) & 0 \\ 0 & I_{k_m} \end{pmatrix} \cdots \begin{pmatrix} \iota_2(z_2) & 0 \\ 0 & I_{k_3+\dots+k_m} \end{pmatrix}$$

shows the surjectivity of f . □

Degenerate density matrices with a diagonal matrix of eigenvalues of the form

$$D_n(\boldsymbol{\lambda}) = \text{diag}_n(\lambda_1 I_{k_1}, \lambda_2 I_{k_2}, \dots, \lambda_m I_{k_m}), \quad I_{k_j} : k_j \times k_j \text{ identity matrix.}$$

are parametrized by

$$(2.17) \quad \Lambda_m^\neq \times G(k_m, \mathbb{C}^n) \times G(k_{m-1}, \mathbb{C}^{n-k_m}) \times \dots \times G(k_2, \mathbb{C}^{n-k_3-\dots-k_m}).$$

We identify $U(k_1 + \dots + k_{j-1})$ and $U(k_1 + \dots + k_{j-1}) \times \{I_{k_j+\dots+k_m}\}$ in $U(n)$. Define the equivalence relation \sim in $U(n)$ by

$$g \sim g' \Leftrightarrow g^{-1}g' \in U(k_1) \times \dots \times U(k_m).$$

Accordingly the elements of

$$U(n)/(U(k_1) \times \dots \times U(k_m)) = U(n)/\sim$$

are the equivalence classes

$$[g] = \{gv; v \in U(k_1) \times \dots \times U(k_m)\}$$

Proposition 2.11. *For $g_j \in U(k_1 + \dots + k_{j-1} + k_j)$ there is a $g_{j-1} \in U(k_1 + \dots + k_{j-1})$ such that*

$$[g_j] = [\iota_j(\pi_j(g_j))g_{j-1}],$$

where $\pi_j(g_j)$ and $[g_{j-1}]$ are uniquely determined by $[g_j]$, for any section ι_j of $U(k_1 + \dots + k_{j-1} + k_j)$ on $U(k_1 + \dots + k_{j-1} + k_j)/(U(k_1 + \dots + k_{j-1}) \times U(k_j))$.

Proof. Since $(U(k_1) \times \cdots \times U(k_{j-1}) \times U(k_j)) \subset U(k_1 + \cdots + k_{j-1}) \times U(k_j)$, $[g_j] = [g'_j]$ implies $\pi_j(g_j) = \pi_j(g'_j)$ for $g_j, g'_j \in U(k_1 + \cdots + k_{j-1} + k_j)$.

It follows from Proposition 2.9 that $g_j = \iota(\pi(g_j))H$ holds for $H = (g_{j-1}, h) \in U(k_1 + \cdots + k_{j-1}) \times U(k_j)$. Thus we have $[g_j] = [\iota(\pi(g_j))H] = [\iota(\pi(g_j))g_{j-1}]$.

Let $[g_j] = [g'_j]$ then there is a $g'_{j-1} \in U(k_1 + \cdots + k_{j-1})$ such that $[g'] = [\iota_j(\pi_j(g))g'_{j-1}]$. Therefore there exists $v \in (U(k_1) \times \cdots \times U(k_{j-1}) \times U(k_j))$ such that $\iota_j(\pi_j(g))g'_{j-1} = \iota(\pi(g))g_{j-1}v$, and $g'_{j-1} = g_{j-1}v$. This shows $[g_{j-1}] = [g'_{j-1}]$. \square

In the same way as Proposition 2.10 one proves the following result.

Proposition 2.12. *For any section ι_j of $U(n - k_{j+1} - \cdots - k_m)$ on $G(k_j, \mathbb{C}^{n-k_{j+1}-\cdots-k_m})$ for $\pi_j : U(n - k_{j+1} - \cdots - k_m) \rightarrow G(k_j, \mathbb{C}^{n-k_{j+1}-\cdots-k_m})$, there is a unique mapping*

$$\phi : U(n)/(U(k_1) \times \cdots \times U(k_m)) \ni [g] \rightarrow$$

$$(z_m, z_{m-1}, \dots, z_2) \in G(k_m, \mathbb{C}^n) \times G(k_{m-1}, \mathbb{C}^{n-k_m}) \times \cdots \times G(k_2, \mathbb{C}^{n-k_3-\cdots-k_m})$$

such that

$$[g] = [g'] \text{ for } g' = \psi(\phi([g])),$$

where

$$\psi(z_m, z_{m-1}, \dots, z_2) = \iota_m(z_m) \begin{pmatrix} \iota_{m-1}(z_{m-1}) & 0 \\ 0 & I_{k_m} \end{pmatrix} \cdots \begin{pmatrix} \iota_2(z_2) & 0 \\ 0 & I_{k_3+\cdots+k_m} \end{pmatrix}.$$

Let

$$\pi : U(n) \ni g \rightarrow [g] \in U(n)/(U(k_1) \times \cdots \times U(k_m)).$$

The above proposition shows that the mapping $\pi \circ \psi$ is surjective. Since ι_j is a section of $U(n - k_{j+1} - \cdots - k_m)$ on $U(n - k_{j+1} - \cdots - k_m)/(U(n - k_j - \cdots - k_m) \times U(k_j))$, and $U(k_1) \times \cdots \times U(k_m) \subset U(n - k_j - \cdots - k_m) \times U(k_j)$, the mapping $\pi \circ \psi$ is injective. Consequently the mappings ϕ and $\pi \circ \psi$ are inverses to each other, and thus we get

Proposition 2.13. *There is a bijection $\phi = (\pi \circ \psi)^{-1}$ between the flag manifold $U(n)/(U(k_1) \times \cdots \times U(k_m))$ and the direct product $G(k_m, \mathbb{C}^n) \times G(k_{m-1}, \mathbb{C}^{n-k_m}) \times \cdots \times G(k_2, \mathbb{C}^{n-k_3-\cdots-k_m})$ of Grassmann manifolds. The mapping $\psi \circ \phi$ is a section of $U(n)$ on $U(n)/(U(k_1) \times \cdots \times U(k_m))$ with respect to π .*

3. PROJECTIVE SPACE

In this section, we summarize the results in Section 2 for $k_j = 1$. The Grassmann manifold $G(1, \mathbb{C}^n)$, the set of all complex 1-dimensional subspaces of \mathbb{C}^n , is called the projective space and denoted by CP^{n-1} .

$$CP^{n-1} = G(1, \mathbb{C}^n) = \{z \in \mathbb{C}^n; z \neq 0\}/(\mathbb{C} \setminus \{0\}).$$

$S_{1,n}$ is a set of permutation σ such that $1 \leq \sigma(1) < \sigma(2) < \cdots < \sigma(n-1) \leq n$ and $1 \leq \sigma(n) \leq n$.

Remark 3.1. Let $\sigma(n) = j$. Then we have $\sigma(k) = k$ if $1 \leq k \leq j-1$ and $\sigma(k) = k+1$ if $j \leq k \leq n-1$. Thus $\sigma \in S_{1,n}$ is characterized by $j = \sigma(n) \in \{1, 2, \dots, n\}$.

Let $j = \sigma(n)$ and $z_j = z_{\sigma(n)}$ is the j -th component of $z \in \mathbb{C}^n$. Define

$$\Omega_j = \{z \in \mathbb{C}^n; z_j \neq 0\}/(\mathbb{C} \setminus \{0\}) \subset CP^{n-1}.$$

Since $z \neq 0$, there is a $j \in \{1, 2, \dots, n\}$ such that $z_j \neq 0$. Thus we have

$$CP^{n-1} = \cup_{j \in \{1, 2, \dots, n\}} \Omega_j.$$

Define the mapping

(3.1)

$$\phi_j = \phi_\sigma : \Omega_\sigma = \Omega_j \ni w \rightarrow (z_{\sigma(1)}, \dots, z_{\sigma(n-1)})/z_{\sigma(n)} = (z_1 \cdots z_{j-1}, z_{j+1}, \dots, z_n)/z_j \in \mathbb{C}^{n-1},$$

where $\sigma(n) = j$. This map ϕ_j gives the homeomorphism

$$\Omega_j \cong \mathbb{C}^{(n-1)}$$

and the set $\{(\Omega_j, \phi_j); j \in \{1, 2, \dots, n\}\}$ an atlas of CP^{n-1} .

There is another parametrization of Ω_j which is more convenient for our purpose.

Introduce

$$B(n-1) = \{x \in \mathbb{C}^{(n-1)}; x^* x < 1\};$$

the set Ω_j can be parametrized by the set $B(n-1)$. We will show this in the following.

Note that $S(1, \mathbb{C}^n) = S(\mathbb{C}^n) = \{z \in \mathbb{C}^n; \|z\| = 1\}$. With

$$\tilde{\Omega}_j = \{z \in S(\mathbb{C}^n); z_j \neq 0\} \subset S(\mathbb{C}^n).$$

one has

$$S(\mathbb{C}^n) = \cup_{j \in \{1, 2, \dots, n\}} \tilde{\Omega}_j.$$

Remark 3.2. Let $\sigma \in S_{1,n}$ such that $\sigma(n) = j$ and denote U_σ by U_j . Then we have $U_j e_k = e_k$ if $1 \leq k \leq j-1$, $U_j e_k = e_{k+1}$ if $j \leq k \leq n-1$ and $U_j e_n = e_j$.

Let π_2 be the surjective mapping

$$\pi_2 : S(\mathbb{C}^n) \ni z \rightarrow w = \text{span } z \in CP^{n-1},$$

where $\text{span } z$ is the complex line spanned by z . If $z, z' \in S(\mathbb{C}^n)$ define the same line, then $z' = ze^{i\theta}$ for some $e^{i\theta} \in U(1)$. Thus we have

$$(3.2) \quad CP^{n-1} \cong S(\mathbb{C}^n)/U(1).$$

In order to parametrize CP^{n-1} , we must choose a unique representative $z \in S(\mathbb{C}^n)$ from (3.2). Note that

$$CP^{n-1} \supset \Omega_j = \{z \in S(\mathbb{C}^n); z_j \neq 0\}/U(1) = \pi_2(\tilde{\Omega}_j).$$

Let $\tilde{\psi}_j$ be the mapping

$$\tilde{\psi}_j : \tilde{\Omega}_j \ni z \rightarrow (z_1 \dots z_{j-1}, z_{j+1} \dots, z_n)^T / e^{i\theta} \in B(n-1), \quad z_j = |z_j| e^{i\theta}.$$

Then $\tilde{\psi}_j$ induces the mapping

$$\psi_j : \Omega_j = \pi_2(\tilde{\Omega}_j) \ni \pi_2(z) \rightarrow (z_1 \dots z_{j-1}, z_{j+1} \dots, z_n)^T / e^{i\theta} \in B(n-1), \quad z_j = |z_j| e^{i\theta}$$

because $\pi_2(z) = \pi_2(z')$ implies $\tilde{\psi}_j(z) = \tilde{\psi}_j(z')$.

Proposition 3.3. *The mapping*

$$\kappa_j : B(n-1) \ni x \rightarrow \pi_2((z_1 \dots z_{j-1}, (1 - x^* x)^{1/2}, z_j \dots, z_{n-1}))^T \in \Omega_j.$$

satisfies $\psi_j \circ \kappa_j = \text{id}$ and $\kappa_j \circ \psi_j = \text{id}$.

Let $x = e_n$. Since $U(n)$ acts on $S(\mathbb{C}^n)$ transitively, and the isotropy group of x is $U(n-1) \times \{1\}$, $U(n)$ acts on $CP^{n-1} = S(\mathbb{C}^n)/U(1)$ transitively by

$$U\pi_2(z) = \pi_2(Uz).$$

Let $y \in CP^{n-1}$ be a complex line of \mathbb{C}^n spanned by the vector e_n . The isotropy group of y is $U(n-1) \times U(1)$, and we have

$$U(n)/(U(n-1) \times U(1)) \cong CP^{n-1}.$$

Let

$$e = e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} O \\ 1 \end{pmatrix} \in S(\mathbb{C}^n)$$

and

$$g = \begin{pmatrix} W & x \\ v & y \end{pmatrix} \in U(n),$$

where $z = \begin{pmatrix} x \\ y \end{pmatrix}$ is the last column of g and $y \in \mathbb{C}^1$.

Define the mapping $\pi_1 : U(n) \rightarrow S(\mathbb{C}^n)$ by

$$\pi_1(g) = ge = \begin{pmatrix} W & x \\ v & y \end{pmatrix} \begin{pmatrix} O \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \in S(\mathbb{C}^n).$$

Suppose that $y \neq 0$. Then there is a unique $e^{i\theta} \in U(1)$ such that $ye^{i\theta} = |y|$. For

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xe^{i\theta} \\ |y| \end{pmatrix}.$$

we have

$$\pi_2 \begin{pmatrix} x \\ y \end{pmatrix} = \pi_2 \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

And with

$$(3.3) \quad g' = W(x') = \begin{pmatrix} (I_{n-1} - x'x'^*)^{1/2} & x' \\ -x'^* & y' \end{pmatrix}.$$

we get

$$\pi_1(g') = \begin{pmatrix} (I_{n-1} - x'x'^*)^{1/2} & x' \\ -x'^* & y' \end{pmatrix} e = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Finally introduce $\pi = \pi_2 \circ \pi_1$ and $\Omega_\sigma = \pi_2(\tilde{\Omega}_\sigma)$. Then we have

$$\pi(g) = \pi(g').$$

Thus we have constructed a local section

$$(3.4) \quad \iota : \Omega_n \ni \pi(g) \rightarrow g' \in U(n)$$

by (3.3).

Proposition 3.4. *For $x \in B(n-1)$ define*

$$(3.5) \quad W(x) = \begin{pmatrix} (I_{n-1} - xx^*)^{1/2} & x \\ -x^* & (1 - x^*x)^{1/2} \end{pmatrix}.$$

Then the mapping

$$\iota_n = W \circ \psi_n : \Omega_n \rightarrow U(n)$$

gives a local section of $U(n)$ on CP^{n-1} for $\pi : U(n) \rightarrow CP^{n-1}$.

Proof. The matrix $W(x)$ in (3.5) is unitary according to Proposition 6.6; the proof of the remaining part is obvious by the above preparations. \square

From now on, we use the notation $\tilde{\Omega}$ for $\tilde{\Omega}_n$, and Ω for Ω_n . Propositions 3.5, 3.3, 3.4 are the special cases ($k = 1$) of Propositions 2.5, 2.2, 2.4 respectively, and we omit the proofs.

Proposition 3.5.

$$(3.6) \quad \tilde{\Omega}_j = U_j \tilde{\Omega}, \text{ and } \Omega_j = U_j \Omega.$$

Next we extend Proposition 3.4 to Ω_j .

Proposition 3.6. *The mapping*

$$\iota_j = U_j W \circ \psi_j : \Omega_j \ni x \rightarrow W(\psi_j(x)) \in U(n)$$

gives a local section of $U(n)$ on $\Omega_j \subset CP^{n-1}$ for $\pi : U(n) \rightarrow CP^{n-1}$.

For $j = 1, \dots, n-1$ introduce

$$(3.7) \quad V_j = \Omega_j \setminus \cup_{i=j+1}^n \Omega_i.$$

Then

$$CP^{n-1} = \cup_{j=1}^n V_j$$

is a disjoint union. We can construct a section $\iota : CP^{n-1} \rightarrow U(n)$ for $\pi : U(n) \rightarrow CP^{n-1}$ as follows.

Definition 3.7. Let $x \in CP^{n-1}$. Define the section $\iota(x)$ by

$$(3.8) \quad \iota(x) = \iota_j(x) \text{ if } x \in V_j.$$

Remark 3.8. Let $\sigma, \sigma' \in S_{1,n}$ be such that $\sigma(n) < \sigma'(n)$. Then $\sigma > \sigma'$ according to the lexicographic ordering. This causes the difference between (3.7) and (2.16)

Proposition 3.9. Let ι be a section of $U(n)$ on CP^{n-1} for π . Then for any $g \in U(n)$, there is a unique $h \in U(n-1) \times U(1)$ such that

$$g = \iota(\pi(g))h.$$

Now we are well prepared to present the parametrization of unitary matrices by the canonical coset decomposition. Symbolically, the canonical coset decomposition is:

$$\begin{aligned} U(n) &= U(n)/(U(n-1) \times U(1)) \cdot U(n-1)/(U(n-2) \times U(1)) \\ &\quad \dots \cdot U(2)/(U(1) \times U(1)) \cdot (U(1) \times \dots \times U(1)). \end{aligned}$$

The following proposition shows the precise meaning of the above formula.

Proposition 3.10. For any section ι_j of $U(j)$ on CP^{j-1} for $\pi_j : U(j) \rightarrow CP^{j-1}$, there is a unique surjection $f : U(n) \ni g \rightarrow (z_n, z_{n-1}, \dots, z_2) \in CP^{n-1} \times CP^{n-2} \times \dots \times CP^1$ and a unique $h \in U(1) \times \dots \times U(1)$ such that

$$g = \iota_n(z_n) \begin{pmatrix} \iota_{n-1}(z_{n-1}) & 0 \\ 0 & I_1 \end{pmatrix} \dots \begin{pmatrix} \iota_2(z_2) & 0 \\ 0 & I_{n-2} \end{pmatrix} h.$$

Nondegenerate density matrices are parametrized by

$$\Lambda_n^\neq \times CP^{n-1} \times CP^{n-2} \times \dots \times CP^1.$$

Let π be the natural map

$$\pi : U(n) \rightarrow U(n)/(U(1) \times \dots \times U(1)).$$

Corollary 3.11. For any section ι_j of $U(j)$ on CP^{j-1} for $\pi_j : U(j) \rightarrow CP^{j-1}$, there is a unique bijection

$$\begin{aligned} \phi : U(n)/(U(1) \times \dots \times U(1)) \ni \pi(g) &\rightarrow \\ (z_n, z_{n-1}, \dots, z_2) &\in CP^{n-1} \times CP^{n-2} \times \dots \times CP^1 \end{aligned}$$

such that

$$\pi(g) = \pi(g') \text{ for } g' = \psi(\phi(\pi(g))),$$

where

$$\psi(z_n, z_{n-1}, \dots, z_2) = \iota_n(z_n) \begin{pmatrix} \iota_{n-1}(z_{n-1}) & 0 \\ 0 & I_1 \end{pmatrix} \dots \begin{pmatrix} \iota_2(z_2) & 0 \\ 0 & I_{n-2} \end{pmatrix}.$$

$\psi \circ \phi$ is a section of $U(n)$ on $U(n)/(U(1) \times \dots \times U(1))$ for π .

4. LOCAL CHARTS OF GRASSMANNIAN

For $F \in S(k, \mathbb{C}^n)$ the submatrix $M(F)_{n-k+1, \dots, n}$ is not necessarily nonsingular, unless $F \in \tilde{\Omega}$. From the uniqueness of the polar decomposition (see [3]) we get

$$(4.1) \quad M(F)_{n-k+1, \dots, n}^* = V |M(F)_{n-k+1, \dots, n}^*|$$

for a unique partial isometry V with null space $N(V) = N(M(F)_{n-k+1, \dots, n}^*)$ (cp. (2.8)). There exists $U \in U(k)$ such that the restriction $U|_{N(V)^\perp}$ of U to the orthogonal complement $N(V)^\perp$ of $N(V)$ is V . Consequently we have

$$M(F)_{n-k+1, \dots, n} = |M(F)_{n-k+1, \dots, n}^*| U^*$$

for some $U^* \in U(k)$. So, we can select some frame $F' = \begin{pmatrix} X' \\ Y' \end{pmatrix} \in S(k, \mathbb{C}^n)$ such that $Y' = M(F)_{n-k+1, \dots, n} U = |M(F)_{n-k+1, \dots, n}^*|$ is a nonnegative operator and $X' = M(F)_{1, \dots, n-k} U$. Since the column vectors of $\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} XU \\ YU \end{pmatrix}$ give an orthonormal frame, we have

$$X'^* X + Y'^2 = (X'^*, Y') \begin{pmatrix} X' \\ Y' \end{pmatrix} = I_k.$$

This shows that $X' \in \bar{B}(n-k, k)$ and $Y' = (I_k - X'^* X')^{1/2}$ where we use the notation

$$\bar{B}(n-k, k) = \{X \in M(n-k, k); X^* X \leq I_k\}.$$

Remark 4.1. We selected a frame F' among the coset $FU(k)$ under the condition that Y' is nonnegative. But this condition can not determine a unique frame F' , because there are many elements $U \in U(k)$ such that the restriction $U|_{N(V)^\perp}$ of U to the orthogonal complement $N(V)^\perp$ of $N(V)$ is V .

Definition 4.2. Define the mapping $\bar{\kappa}_e : \bar{B}(n-k, k) \rightarrow G(k, \mathbb{C}^n)$ by

$$\bar{\kappa}_e : \bar{B}(n-k, k) \ni X \rightarrow \pi_2 \left(\begin{pmatrix} X \\ (I_k - X^* X)^{1/2} \end{pmatrix} \right) \in G(k, \mathbb{C}^n).$$

Remark 4.3. The mapping

$$\kappa_e : B(n-k, k) \ni X \rightarrow \pi_2 \left(\begin{pmatrix} X \\ (I_k - X^* X)^{1/2} \end{pmatrix} \right) \in \Omega.$$

of Proposition 2.2 is bijective. But the mapping $\bar{\kappa}_e$ of the above definition is not bijective but only surjective.

For $X \in \bar{B}(n-k, k)$ denote

$$W(X) = \begin{pmatrix} (I - XX^*)^{1/2} & X \\ -X^* & Y \end{pmatrix}, \quad Y = (I - X^* X)^{1/2}.$$

Proposition 4.4. For any $g \in U(n)$ we can find $X \in \bar{B}(n-k, k)$, $V \in U(k)$ and $h \in U(n-k) \times I_k$ such that

$$\begin{aligned} g &= \begin{pmatrix} (I - XX^*)^{1/2} & X \\ -X^* & Y \end{pmatrix} \begin{pmatrix} I_{n-k} & O \\ O & V \end{pmatrix} h \\ &= \begin{pmatrix} (I - XX^*)^{1/2} & X \\ -X^* & Y \end{pmatrix} \begin{pmatrix} U & O \\ O & V \end{pmatrix}, \end{aligned}$$

with $U \in U(n-k)$ and $\bar{B}(m, k) = \{X \in M(m, k); X^* X \leq I_k\}$.

Remark 4.5. The above proposition is the counterpart of Proposition 2.9. Note that $V \in U(k)$ is not unique, and consequently, X and U are also not unique. If we use $B(n-k, k)$ instead of $\bar{B}(n-k, k)$, then we have the uniqueness. The above proposition only holds if g satisfies $\pi(g) \in \Omega$. The question of uniqueness is addressed in the following proposition.

Proposition 4.6. *Let $g, g' \in U(n-k_1-\dots-k_j)$ and $W(X_j), W(X'_j)$ for $X_j, X'_j \in \bar{B}(n-k_1-\dots-k_j, k_j)$ such that $W(X_j)g = W(X'_j)g'$. Then $X_j = X'_j$ and $g = g'$.*

Proof. Let

$$\pi_1 : U(n-k_1-\dots-k_j) \rightarrow S(k_j, \mathbb{C}^{n-k_1-\dots-k_j})$$

be the projection defined by (2.9). Then $\begin{pmatrix} X_j \\ Y_j \end{pmatrix} = \pi(W(X_j)g) = \pi(W(X'_j)g') = \begin{pmatrix} X'_j \\ Y'_j \end{pmatrix}$, and $X_j = X'_j$, $g = W(X_j)^{-1}W(X'_j)g' = g'$.

It follows from Proposition 4.14 that for any $U_1 \in U(n-k_1)$, there is $W(X_2)$ for $X_2 \in \bar{B}(n-k_1-k_2, k_2)$ such that

$$U_1 = W(X_2) \begin{pmatrix} U_2 & O \\ O & V_2 \end{pmatrix} = \begin{pmatrix} (I - X_2 X_2^*)^{1/2} U_2 & X_2 V_2 \\ -X_2^* U_2 & Y_2 V_2 \end{pmatrix},$$

where $U_2 \in U(n-k_1-k_2)$ and $V_2 \in U(k_2)$. Then (5) and (6) imply, for $g \in U(n)$,

$$\begin{aligned} g &= W(X_1) \begin{pmatrix} \begin{pmatrix} (I - X_2 X_2^*)^{1/2} U_2 & X_2 V_2 \\ -X_2^* U_2 & Y_2 V_2 \end{pmatrix} & O \\ O & V_1 \end{pmatrix} \\ &= W(X_1) \begin{pmatrix} \begin{pmatrix} (I - X_2 X_2^*)^{1/2} & X_2 \\ -X_2^* & Y_2 \end{pmatrix} & O \\ O & I_{k_1} \end{pmatrix} \begin{pmatrix} U_2 & O & O \\ O & V_2 & O \\ O & O & V_1 \end{pmatrix}. \end{aligned}$$

□

Iteration of this procedure gives

Proposition 4.7. (1) *For any $g \in U(n)$ there exists $(X_m, \dots, X_2) \in \bar{B}(n-k_m, k_m) \times \dots \times \bar{B}(n-k_3-\dots-k_m, k_2)$ and $(V_1, \dots, V_m) \in U(k_1) \times \dots \times U(k_m)$ such that*

$$g = W(X_m) \begin{pmatrix} W(X_{m-1}) & O \\ O & I_{k_m} \end{pmatrix} \dots \begin{pmatrix} W(X_2) & O \\ O & I_{k_3+\dots+k_m} \end{pmatrix} \begin{pmatrix} V_1 & \dots & O \\ \vdots & \ddots & \vdots \\ O & \dots & V_m \end{pmatrix}.$$

(2) *The mapping*

$$\begin{aligned} &\bar{B}(n-k_m, k_m) \times \dots \times \bar{B}(n-k_3-\dots-k_m, k_2) \ni (X_m, \dots, X_2) \rightarrow \\ &W(X_m) \begin{pmatrix} W(X_{m-1}) & O \\ O & I_{k_m} \end{pmatrix} \dots \begin{pmatrix} W(X_2) & O \\ O & I_{k_3+\dots+k_m} \end{pmatrix} \in U(n) \end{aligned}$$

is injective.

Proof. The existence of X_j and V_j follows from Proposition 4.14 and the injectivity of the mapping follows from Proposition 4.6. □

Remark 4.8. The above proposition is the counterpart of Proposition 2.10 and the following proposition is the counterpart of Proposition 3.11.

Proposition 4.9. *Let $x_j \in \bar{B}(j-1) = \{x \in \mathbb{C}^{j-1}; \|x\| \leq 1\}$ and*

$$(4.2) \quad W(x_j) = \begin{pmatrix} (I_{j-1} - x_j x_j^*)^{1/2} & x_j \\ -x_j^* & y_j \end{pmatrix}, \quad y_j = (1 - x_j^* x_j)^{1/2}.$$

(1) For any $g \in U(n)$ there exists $(x_n, x_{n-1}, \dots, x_2) \in \bar{B}(n-1) \times \bar{B}(n-2) \times \dots \times \bar{B}(1)$ and $(V_1, \dots, V_n) \in U(1) \times \dots \times U(1)$ such that

$$(4.3) \quad g = W(x_n) \begin{pmatrix} W(x_{n-1}) & O \\ O & I_1 \end{pmatrix} \cdots \begin{pmatrix} W(x_2) & O \\ O & I_{n-2} \end{pmatrix} \begin{pmatrix} V_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & V_n \end{pmatrix}.$$

(2) The mapping

$$\begin{aligned} & \bar{B}(n-1) \times \dots \times \bar{B}(1) \ni (x_n, \dots, x_2) \rightarrow \\ & W(x_n) \begin{pmatrix} W(x_{n-1}) & O \\ O & I_1 \end{pmatrix} \cdots \begin{pmatrix} W(x_2) & O \\ O & I_{n-2} \end{pmatrix} \in U(n) \end{aligned}$$

is injective.

Instead of $\bar{B}(j-1)$ in [7, 9] the following parameter space \bar{Q}_j

$$(4.4) \quad \bar{Q}_j = \{(\theta_j, \zeta_j); 0 \leq \theta \leq \pi/2, \zeta_j \in S(\mathbb{C}^{j-1})\}, \quad S(\mathbb{C}^{j-1}) = \{\zeta \in \mathbb{C}^{j-1}; \|\zeta\| = 1\}$$

is introduced for the Jarlskog parametrization [11], [12]. The mapping

$$\bar{Q}_j \ni (\theta_j, \zeta_j) \rightarrow \sin \theta_j \zeta_j = x \in \bar{B}(j-1) = \{x \in \mathbb{C}^{j-1}; \|x\| \leq 1\}$$

shows that both parameter spaces are the same.

The formula in [11] which correspond to the formula (4.3) is

$$U_n = A_{n,n} A_{n,n-1} \cdots A_{n,2} D(e^{i\alpha_1}, \dots, e^{i\alpha_n}),$$

where, using bra and ket notation of Physics,

$$A_{n,j} = \begin{pmatrix} V_j(\theta_j, \zeta_j) & 0 \\ 0 & I_{n-j} \end{pmatrix}, \quad V_j(\theta_j, \zeta_j) = \begin{pmatrix} I_{j-1} - (1 - \cos \theta_j) |\zeta_j\rangle \langle \zeta_j| & \sin \theta_j |\zeta_j\rangle \\ -\sin \theta_j \langle \zeta_j| & \cos \theta_j \end{pmatrix}.$$

For $x = \sin \theta_j \zeta_j$, $V_j(\theta_j, \zeta_j)$ and $W(x)$ of (4.2) are precisely the same.

In [1] there is a statement that a typical coset representative in the coset space $U(2)/(U(1) \times U(1))$ is

$$V = \begin{pmatrix} \cos \alpha & e^{i\phi} \sin \alpha \\ -e^{-i\phi} \sin \alpha & \cos \alpha \end{pmatrix}.$$

V is $V_1(\theta_1, \zeta_1)$ for $(\theta_1, \zeta_1) = (\alpha, \phi) \in \bar{Q}_1$. But if $\cos \alpha = 0$ and $e^{i\phi} \neq 1$,

$$\begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}.$$

This is an example showing that the expression of Proposition 4.14 is not unique. We find

$$\begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix}^* = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^*,$$

which shows that the density matrix $\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}$ has two parametrization, i.e., $((\lambda, \mu), 1)$ and $((\lambda, \mu), e^{i\alpha})$.

However, if we use the parameter space

$$Q_j = \{(\theta_j, \zeta_j); 0 \leq \theta < \pi/2, \zeta_j \in S(\mathbb{C}^{j-1})\}, \quad S(\mathbb{C}^{j-1}) = \{\zeta \in \mathbb{C}^{j-1}; \|\zeta\| = 1\}$$

instead of \bar{Q}_j of (4.4), then the uniqueness is recovered. But not every g can not be expressed by such parameters (see Remark 4.6). Let $g = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix}$. Then $\pi(g) = \pi_2 \begin{pmatrix} e^{i\phi} \\ 0 \end{pmatrix}$ does not belong to $\Omega = \Omega_2$ but belongs to $\Omega_1 = \pi_2(\tilde{\Omega}_1)$, where $\tilde{\Omega}_j = \{z = (z_1, z_2) \in S(\mathbb{C}^2); z_j \neq 0\}$. The unitary matrix U_1 in Remark 3.2 is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the mapping κ_1 which gives bijection between $B(1)$ and Ω_1 is $\kappa_1 = U_1\kappa$:

$$\kappa_1 : B(1) \ni 0 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pi_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pi_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pi_2 \begin{pmatrix} e^{i\phi} \\ 0 \end{pmatrix},$$

and

$$\lambda_1 = U_1\lambda : B(1) \ni 0 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \iota_1(\pi(g)).$$

Thus g has the unique form

$$g = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} = \iota_1(\pi(g))h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e^{-i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

It follows from Remarks 4.3 and 4.6 that Proposition 4.14 for the parameter space $B(n-k, k)$ is valid only if $\pi(g) \in \Omega$. As shown in the following, $G(k, \mathbb{C}^n) \setminus \Omega = \partial\Omega$. So, in most cases $\partial\Omega_\sigma$ is negligible.

Definition 4.10. Let $(\Omega_\sigma, \phi_\sigma)_{\sigma \in S}$ be the system of (at most) countable coordinate neighborhoods of a m -dimensional manifold M . A subset $A \subset M$ is said to have the measure zero if for every coordinate neighborhood $(\Omega_\sigma, \phi_\sigma)$ the set $\phi_\sigma(A \cap \Omega_\sigma)$ has Lebesgue measure zero in \mathbb{R}^m .

Proposition 4.11. Let $(\Omega_\sigma, \phi_\sigma)_{\sigma \in S}$ be the atlas of the Grassmann manifold $G(k, \mathbb{C}^n)$. Then $\Omega_\sigma \cap \Omega_{\sigma'}$ is an open and dense subset of Ω_σ .

Proof. Assume $\sigma \neq \sigma'$. Let

$$M_\sigma M_{\sigma(n-k+1), \dots, \sigma(n)}^{-1} = \begin{pmatrix} Z \\ I \end{pmatrix} = \begin{pmatrix} z_{11} & \dots & z_{1k} \\ \vdots & \ddots & \vdots \\ z_{n-k,1} & \dots & z_{n-k,k} \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} = N.$$

Define a function $f(Z)$ of $Z \in M(n-k, k)$ by

$$f(Z) = \det N_{\sigma^{-1} \circ \sigma'(n-k+1), \sigma^{-1} \circ \sigma'(n-k+2), \dots, \sigma^{-1} \circ \sigma'(n)}.$$

Then $f(Z)$ is a non-constant polynomial (provided $\sigma \neq \sigma'$), and

$$\Omega_\sigma \supset \Omega_\sigma \cap \Omega_{\sigma'} = \{Z \in M(n-k, k); f(Z) \neq 0\},$$

$$\Omega_\sigma \setminus \Omega_{\sigma'} = \{Z \in M(n-k, k); f(Z) = 0\}.$$

Since $f(Z)$ is a polynomial of Z , $f(Z)$ is continuous and $\Omega_\sigma \cap \Omega_{\sigma'}$ is open subset of Ω_σ . For any neighborhood V of $Z_0 \in \Omega_\sigma \setminus \Omega_{\sigma'}$ there exists $Z_1 \in V$ such that $Z_1 \in \Omega_\sigma \cap \Omega_{\sigma'}$ ($\Omega_\sigma \cap \Omega_{\sigma'}$ is dense in Ω_σ). Otherwise, there exists a neighborhood V of Z_0 such that $f(Z) = 0$ on V . Since $f(Z)$ is an analytic function, we have $f(Z) \equiv 0$. This is a contradiction. \square

Corollary 4.12. Ω_σ is an open and dense subset of $G(k, \mathbb{C}^n)$, and therefore $G(k, \mathbb{C}^n) \setminus \Omega_\sigma = \partial\Omega_\sigma$ and $\partial\Omega_\sigma$ is a set of measure zero.

Proof. This follows from Relation (2.1). \square

Note that $(U(n), G(k, \mathbb{C}^n), \pi, U(n-k) \times U(k))$ has the structure of a fiber bundle, where $U(n)$, $G(k, \mathbb{C}^n)$, and $U(n-k) \times U(k)$ are the total space, the base space, and the fiber respectively, and $\pi : U(n) \rightarrow G(k, \mathbb{C}^n)$ is a continuous surjection satisfying a local triviality condition: For every $z \in G(k, \mathbb{C}^n)$, there is an open neighborhood Ω_σ of z (which will be called a trivializing neighborhood) such that there is a homeomorphism

$$\phi : \Omega_\sigma \times (U(n-k) \times U(k)) \ni (z, h) \rightarrow \phi(z, h) = \iota_\sigma(z)h \in \pi^{-1}(\Omega_\sigma).$$

Proposition 4.13. $\pi^{-1}(\partial\Omega_e)$ is a set of measure zero.

Proof. Since $\partial\Omega_e \cap \Omega_\sigma$ is the boundary of $\Omega_e \cap \Omega_\sigma$ in Ω_σ , $\partial\Omega_e \cap \Omega_\sigma \times (U(n-k) \times U(k)) \cong \pi^{-1}(\partial\Omega_e \cap \Omega_\sigma)$ is the boundary of $\Omega_e \cap \Omega_\sigma \times (U(n-k) \times U(k)) \cong \pi^{-1}(\Omega_e \cap \Omega_\sigma)$ in $\Omega_\sigma \times (U(n-k) \times U(k)) \cong \pi^{-1}(\Omega_\sigma)$. Thus $\pi^{-1}(\partial\Omega_e)$ is the boundary of $\pi^{-1}(\Omega_e)$ in $U(n) = \pi^{-1}(G(k, \mathbb{C}^n))$ and a set of measure zero. \square

Proposition 4.14. For almost all $g \in U(n)$, there is a unique $X \in B(n-k, k)$ and $h \in U(n-k) \times U(k)$ such that

$$g = W(X)h.$$

Proof. The proposition follows from the fact that $\pi^{-1}(\partial\Omega_e) \cup \pi^{-1}(\Omega_e) = U(n)$ and the previous proposition. \square

Proposition 4.15. For almost all $g \in U(n)$ we have a mapping $U(n) \ni g \rightarrow (X_3, X_2) \in B(n-k_3, k_3) \times B(n-k_3-k_2, k_2)$ and a unique $h \in U(k_1) \times U(k_1) \times U(k_3)$ such that

$$(4.5) \quad g = W(X_3) \begin{pmatrix} W(X_2) & O \\ O & I_{k_3} \end{pmatrix} h.$$

Proof. Let $m = n - k_3$ and consider the two fiber bundles $F = (U(n), G(k_3, \mathbb{C}^n), \pi, U(n-k_3) \times U(k_3))$ and $F' = (U(m), G(k_2, \mathbb{C}^m), \pi', U(m-k_2) \times U(k_2))$. Let Ω'_e be a trivializing neighborhood of F' . Then $\partial\Omega'_e \times (U(m-k_2) \times U(k_2)) \cong \pi'^{-1}(\partial\Omega'_e)$ is the boundary of $\Omega'_e \times (U(m-k_2) \times U(k_2)) \cong \pi'^{-1}(\Omega'_e)$ in $U(m)$ and $\Omega_e \times \pi'^{-1}(\partial\Omega'_e) \times U(k_3) \cong \phi(\Omega_e, \pi'^{-1}(\partial\Omega'_e) \times U(k_3))$ is the boundary of $\Omega_e \times \pi'^{-1}(\Omega'_e) \times U(k_3) \cong \phi(\Omega_e, \pi'^{-1}(\Omega'_e) \times U(k_3))$ in $\Omega_e \times U(m) \times U(k_3) \cong \pi^{-1}(\Omega_e)$. Thus $\phi(\Omega_e, \pi'^{-1}(\partial\Omega'_e) \times U(k_3))$ is a set of measure zero. Since

$$\phi(\Omega_e, \pi'^{-1}(\Omega'_e) \times U(k_3)) \cup \phi(\Omega_e, \pi'^{-1}(\partial\Omega'_e) \times U(k_3)) = \pi^{-1}(\Omega_e),$$

almost all $g \in U(n)$ are expressed as $y = \phi(\Omega_e, \pi'^{-1}(\Omega'_e) \times U(k_3))$, which is just (4.5). \square

Continuation of the above chain of arguments implies the following theorem.

Theorem 4.16. For almost all $g \in U(n)$ we have a mapping $U(n) \ni g \rightarrow (X_m, \dots, X_2) \in B(n-k_m, k_m) \times \dots \times B(n-k_3 - \dots - k_m, k_2)$ and a unique $h \in U(k_1) \times \dots \times U(k_m)$ such that

$$g = W(X_m) \begin{pmatrix} W(X_{m-1}) & O \\ O & I_{k_m} \end{pmatrix} \dots \begin{pmatrix} W(X_2) & O \\ O & I_{k_3 + \dots + k_m} \end{pmatrix} h.$$

5. SECTION ON GRASSMANNIAN

The mapping

$$W(X) = \begin{pmatrix} (I_{n-k} - XX^*)^{1/2} & X \\ -X^* & (I_k - X^*X)^{1/2} \end{pmatrix}$$

of (2.13) for $X \in B(n-k, k)$ which gives a local section $\iota_e = W \circ \psi_e : \Omega_e \rightarrow U(n)$ is not really suitable for concrete calculations. So here we construct a local section using simpler ones

$$W(x) = \begin{pmatrix} (I_{n-1} - xx^*)^{1/2} & x \\ -x^* & (1 - x^*x)^{1/2} \end{pmatrix}.$$

of (3.5) for $x \in B(n-1)$ which gives a local section $\iota_n = W \circ \psi_n : \Omega_n \rightarrow U(n)$.

We begin with the embedding of CP^{m-1} in CP^{n-1} ($m < n$). $S(\mathbb{C}^m)$ can be embedded in $S(\mathbb{C}^n)$ by

$$\iota_{n,m} : S(\mathbb{C}^m) \ni (z_1, \dots, z_m)^T \rightarrow (z_1, \dots, z_m, 0, \dots, 0)^T \in S(\mathbb{C}^n).$$

This mapping $\iota_{n,m}$ induces the embedding $\iota_{n,m} : CP^{m-1} \rightarrow CP^{n-1}$.

Let $\sigma_{j,n} \in S_{1,n}$ such that $\sigma_{j,n}(n) = j$. Then $\tilde{\Omega}_m = \tilde{\Omega}_{m,m} = \tilde{\Omega}_{\sigma_{m,m}}$ (resp. $\Omega_m = \Omega_{m,m} = \Omega_{\sigma_{m,m}}$) is identified with $\tilde{\Omega}_{m,n} = \tilde{\Omega}_{\sigma_{m,n}}$ (resp. $\Omega_{m,n} = \Omega_{\sigma_{m,n}}$). Let $\tilde{\psi}_{m,n}$ be the mapping

$$\tilde{\psi}_{m,n} : \tilde{\Omega}_{m,n} \ni z = (z_1, \dots, z_{m-1}, 0, \dots, 0, z_m)^T \rightarrow (z_1, \dots, z_{m-1}, 0, \dots, 0)^T / e^{i\theta} \in B(n-1), \quad z_m = |z_m|e^{i\theta}.$$

Then $\tilde{\psi}_{m,n}$ induces the mapping

$$\psi_{m,n} : \Omega_{m,n} \ni \pi_2(z) \rightarrow (z_1, \dots, z_{m-1}, 0, \dots, 0)^T / e^{i\theta} \in B(n-1), \quad z_m = |z_m|e^{i\theta},$$

where $\pi_2 : S(\mathbb{C}^m) \rightarrow CP^{m-1}$ is the canonical projection.

Proposition 5.1. *Let*

$$S_k(\mathbb{C}^n) \ni F = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad X \in M(n-k, k), \quad Y \in M(k, k)$$

be given with $\det Y \neq 0$. Then there exists a unique $U \in U(k)$ such that $Y' = YU = T$ where T is a lower triangular matrix:

$$(5.1) \quad T = \begin{pmatrix} t_{11} & 0 & \dots & 0 \\ t_{21} & t_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & t_{kk} \end{pmatrix}, \quad t_{jj} > 0, \quad 1 \leq j \leq k.$$

Proof. Let \mathbb{C}^{*k} be the set of all complex row k vectors $\mathbf{z} = (z_1, \dots, z_k)$ with an inner product $(\mathbf{z}, \mathbf{z}') = \sum_{j=1}^k z_j \bar{z}'_j$, and $\mathbf{z}^* = (\bar{z}_1, \dots, \bar{z}_k)^T$. Let \mathbf{y}_j be the j -th row vector of Y and $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be the Schmidt's orthogonalization of $\{\mathbf{y}_1, \dots, \mathbf{y}_k\}$, i.e., $\mathbf{u}_1 = \mathbf{y}_1 / \|\mathbf{y}_1\|$, $\mathbf{u}_j = \mathbf{x}_j / \|\mathbf{x}_j\|$, $\mathbf{x}_j = \mathbf{y}_j - \sum_{i=1}^{j-1} (\mathbf{y}_j, \mathbf{u}_i) \mathbf{u}_i$. Define $U \in U(k)$ by

$$U = (\mathbf{u}_1^*, \dots, \mathbf{u}_k^*).$$

Since $\text{span}\{\mathbf{y}_1, \dots, \mathbf{y}_j\} = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_j\}$, $(\mathbf{y}_i, \mathbf{u}_i) = 0$ if $i < j$, and $(\mathbf{y}_j, \mathbf{u}_j) > 0$. Thus we have $YU = T$. For the uniqueness of U , suppose $Y = TU^* = T'U'^*$ for

$$U^* = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_k \end{pmatrix}, \quad U'^* = \begin{pmatrix} \mathbf{u}'_1 \\ \mathbf{u}'_2 \\ \vdots \\ \mathbf{u}'_k \end{pmatrix}, \quad T' = \begin{pmatrix} t'_{11} & 0 & \dots & 0 \\ t'_{21} & t'_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & t'_{kk} \end{pmatrix}.$$

Then we have $\mathbf{y}_1 = t_{11}\mathbf{u}_1 = t'_{11}\mathbf{u}'_1$ and therefore $t_{11} = t'_{11}$ and $\mathbf{u}_1 = \mathbf{u}'_1$. From $\mathbf{y}_2 = t_{21}\mathbf{u}_1 + t_{22}\mathbf{u}_2 = t'_{21}\mathbf{u}_1 + t'_{22}\mathbf{u}'_2$, we have $t_{21} = (\mathbf{y}_2, \mathbf{u}_1) = t'_{21}$ and $\mathbf{y}_2 - t_{21}\mathbf{u}_1 = t_{22}\mathbf{u}_2 = t'_{22}\mathbf{u}'_2$. Thus we have $t_{22} = t'_{22}$ and $\mathbf{u}_2 = \mathbf{u}'_2$. Continuing these procedures, we get $U^* = U'^*$, i.e., the uniqueness of U . \square

Proposition 5.2. *Let $g_n \in U(n)$ be given with*

$$(5.2) \quad g_n = \begin{pmatrix} W & X \\ V & T \end{pmatrix},$$

where $W \in M(n-k, n-k)$, $X \in M(n-k, k)$, $V \in M(k, n-k)$ and $T \in M(k, k)$ is a lower triangular matrix with positive diagonal elements as (5.1). Then there exist $x \in \mathbb{C}^{n-1}$ and $g_{n-1} \in U(n-1)$ such that

$$(5.3) \quad g_n = W(x) \cdot (g_{n-1} \times I_1),$$

where

$$W(x) = \begin{pmatrix} (I_{n-1} - xx^*)^{1/2} & x \\ -x^* & (1 - x^*x)^{1/2} \end{pmatrix},$$

and g_{n-1} has the form

$$(5.4) \quad g_{n-1} = \begin{pmatrix} W' & X' \\ V' & T' \end{pmatrix},$$

where $W' \in M(n-k, n-k)$, $X' \in M(n-k, k-1)$, $V' \in M(k-1, n-k)$ and $T' \in M(k-1, k-1)$ is a lower triangular matrix with positive diagonal elements of the form

$$(5.5) \quad T = \begin{pmatrix} T' & 0 \\ * & t_{kk} \end{pmatrix}.$$

Proof. We only have to show that $T' \in M(k-1, k-1)$ is of the form of (5.5). Let $(x, t)^T$ ($x \in \mathbb{C}^{n-1}$, $t \in \mathbb{C}$) be the last column of the matrix g_n .

Since T is lower triangular, $x = (x_1, \dots, x_{n-k}, 0, \dots, 0)^T$, $(I_{n-1} - xx^*)^{1/2} = (I_{n-k} - x'x'^*)^{1/2} \times I_{k-1}$, where $x' = (x_1, \dots, x_{n-k})^T$. Since $W(x)^{-1}g_n = g_{n-1} \times I_1$,

$$T' \times I_1 = \begin{pmatrix} O_1 & I_{k-1} & O_2 \\ x^* & O_3 & (1 - x^*x)^{1/2} \end{pmatrix} \begin{pmatrix} X \\ T \end{pmatrix},$$

where O_1 (resp. O_2, O_3) is the $(k-1) \times (n-k)$ (resp. $(k-1) \times 1$, $1 \times (k-1)$) zero matrix. Therefore

$$\begin{pmatrix} T' & O_4 \end{pmatrix} = \begin{pmatrix} I_{k-1} & O_2 \end{pmatrix} T,$$

where O_4 is the $(k-1) \times 1$ zero matrix and T' is the $(k-1) \times (k-1)$ -submatrix of T . \square

Proposition 5.3. *Let $\pi : U(n) \rightarrow G(k, \mathbb{C}^n)$ be the canonical projection. Then there is a unique surjection*

$$f : U(n) \supset \pi^{-1}(\Omega_e) \ni g \rightarrow (z_k, z_{k-1}, \dots, z_1)$$

$$\in \Omega_{m,n} \times \Omega_{m,n-1} \times \dots \times \Omega_{m,n-k+1} \subset CP^{n-1} \times CP^{n-2} \times \dots \times CP^{n-k}$$

for $m = n - k + 1$ and $\Omega_e \subset G(k, \mathbb{C}^n)$, and a unique $h \in U(n-k) \times U(k)$ such that

$$(5.6) \quad g = W(\psi_{m,n}(z_k)) \begin{pmatrix} W(\psi_{m,n-1}(z_{k-1})) & 0 \\ 0 & I_1 \end{pmatrix} \dots \begin{pmatrix} W(\psi_{m,n-k+1}(z_1)) & 0 \\ 0 & I_{k-1} \end{pmatrix} h.$$

Proof. Since $g \in \pi^{-1}(\Omega_e)$, g has the form of (5.2) with $\det Y \neq 0$ and there exists $U \in U(k)$ such that $T = YU$ has the form of (5.1). Let

$$g_n = g \begin{pmatrix} I_{n-k} & 0 \\ 0 & U \end{pmatrix}.$$

Then it follows from Proposition 5.2 that there exists $g_{n-1} \in U(n-1)$ which satisfies (5.3) and has the form of (5.4) where T' again satisfies the hypothesis of Proposition 5.2. Iterating this argument, we get

$$g_n = W(\psi_{m,n}(z_k)) \begin{pmatrix} W(\psi_{m,n-1}(z_{k-1})) & 0 \\ 0 & I_1 \end{pmatrix} \dots \begin{pmatrix} W(\psi_{m,n-k+1}(z_1)) & 0 \\ 0 & I_{k-1} \end{pmatrix} h,$$

where $h = g_{n-k} \times I_k$, $g_{n-k} \in U(n-k)$. This shows (5.6) with $h = g_{n-k} \times U$. The relation $f(g') = (z_k, z_{k-1}, \dots, z_1)$ for $(z_k, z_{k-1}, \dots, z_1) \in \Omega_{m,n} \times \Omega_{m,n-1} \times \dots \times \Omega_{m,n-k+1}$ and

$$g' = W(\psi_{m,n}(z_k)) \begin{pmatrix} W(\psi_{m,n-1}(z_{k-1})) & 0 \\ 0 & I_1 \end{pmatrix} \dots \begin{pmatrix} W(\psi_{m,n-k+1}(z_1)) & 0 \\ 0 & I_{k-1} \end{pmatrix}$$

show the surjectivity of f . \square

Corollary 5.4. *Let $\pi : U(n) \rightarrow G(k, \mathbb{C}^n)$ be the canonical projection, and ι_j the section of $U(j)$ on CP^{j-1} defined by (3.8). Then there is a unique bijection*

$$\phi_e : G(k, \mathbb{C}^n) \supset \Omega_e \ni \pi(g) \rightarrow$$

$$(z_k, z_{k-1}, \dots, z_1) \in \Omega_{m,n} \times \Omega_{m,n-1} \times \dots \times \Omega_{m,n-k+1} \subset CP^{n-1} \times CP^{n-2} \times \dots \times CP^{n-k}$$

such that

$$\pi(g) = \pi(g') \text{ for } g' = \psi(\phi_e(\pi(g))),$$

where

$$\psi(z_k, z_{k-1}, \dots, z_1) = \iota_n(z_k) \begin{pmatrix} \iota_{n-1}(z_{k-1}) & 0 \\ 0 & I_1 \end{pmatrix} \cdots \begin{pmatrix} \iota_{n-k+1}(z_1) & 0 \\ 0 & I_{k-1} \end{pmatrix}.$$

$\psi \circ \phi_e$ is a section of $U(n)$ on $\Omega_e \subset G(k, \mathbb{C}^n)$ for π .

6. LIE ALGEBRAIC BACK GROUND

In the articles which we have mentioned many statements are based on the use of the Lie algebra $\mathfrak{u}(n)$ of Lie group $U(n)$. We comment here on the connection with the approach presented above.

The Lie algebra $\mathfrak{u}(n)$ of the Lie group $U(n)$ is defined by

$$\mathfrak{u}(n) = \{X \in M(n, n); \forall t \in \mathbb{R}, \exp tX \in U(n)\}.$$

From the relation

$$\exp tX^* = (\exp tX)^* = (\exp tX)^{-1} = \exp -tX$$

it follows

$$\mathfrak{u}(n) = \{X \in M(n, n); X^* = -X\}.$$

Let $n = k_1 + k_2$. Then the Lie algebra of the Lie group $U(k_1) \times U(k_2)$ is $\mathfrak{u}(k_1) \oplus \mathfrak{u}(k_2)$, namely, the set of the elements of the form

$$\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \quad X_j \in \mathfrak{u}(k_j).$$

Let \mathfrak{p} be a subset of $\mathfrak{u}(n)$ such that

$$\mathfrak{u}(n) = \mathfrak{u}(k_1) \oplus \mathfrak{u}(k_2) \oplus \mathfrak{p}.$$

Then \mathfrak{p} consists of the elements of the form

$$K = K(B) = \begin{pmatrix} O_1 & B \\ -B^* & O_2 \end{pmatrix},$$

where O_j ($j = 1, 2$) is the $k_j \times k_j$ matrix whose entries are all zero and B is an $k_1 \times k_2$ complex matrix. Since the space

$$\mathfrak{u}(n)/(\mathfrak{u}(k_1) \oplus \mathfrak{u}(k_2)) \cong \mathfrak{p} = \{K(B); B \in M(k_1, k_2)\}$$

is considered to be the tangent space of the homogeneous space $U(n)/(U(k_1) \times U(k_2))$ at $o = \pi(e)$, where e is the identity of $U(n)$ and $\pi : U(n) \rightarrow U(n)/(U(k_1) \times U(k_2))$ of (2.11), we study $\exp K(B)$. First, we have

$$K^2 = \begin{pmatrix} -BB^* & O \\ O^* & -B^*B \end{pmatrix},$$

where BB^* is an $k_1 \times k_1$ -matrix, B^*B an $k_2 \times k_2$ -matrix and O is the $k_1 \times k_2$ -matrix whose entries are all zero. Observe now

$$\begin{aligned} K^{2n+2} &= \begin{pmatrix} -BB^* & O \\ O^* & -B^*B \end{pmatrix}^{n+1} = \begin{pmatrix} -\sqrt{BB^*}^2 & O \\ O^* & -\sqrt{B^*B}^2 \end{pmatrix}^{n+1} \\ &= \begin{pmatrix} (-1)^{n+1} \sqrt{BB^*}^{2n+2} & O \\ O^* & (-1)^{n+1} \sqrt{B^*B}^{2n+2} \end{pmatrix}. \end{aligned}$$

This gives

$$I_n + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} K^{2n+2} = \begin{pmatrix} \cos \sqrt{BB^*} & O \\ O^* & \cos \sqrt{B^*B} \end{pmatrix}$$

and similarly

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} K^{2n} K &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} (-BB^*)^n & O \\ O^* & (-B^*B)^n \end{pmatrix} \begin{pmatrix} O_1 & B \\ -B^* & O_2 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & (-BB^*)^n B \\ (-B^*B)^n (-B^*) & O_2 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & B(-B^*B)^n \\ (-B^*B)^n (-B^*) & O_2 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & B(-1)^n \sqrt{B^*B}^{2n} \\ (-1)^n \sqrt{B^*B}^{2n} (-B^*) & O_2 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & B\sqrt{B^*B}^{-1} (-1)^n \sqrt{B^*B}^{2n+1} \\ \sqrt{B^*B}^{-1} (-1)^n \sqrt{B^*B}^{2n+1} (-B^*) & O_2 \end{pmatrix} \\
&= \begin{pmatrix} O_1 & B\sqrt{B^*B}^{-1} \sin \sqrt{B^*B} \\ \sqrt{B^*B}^{-1} \sin \sqrt{B^*B} (-B^*) & O_2 \end{pmatrix}.
\end{aligned}$$

Thus we conclude

$$e^K = I_n + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} K^{2n+2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} K^{2n+1} = \begin{pmatrix} \cos \sqrt{BB^*} & B \frac{\sin \sqrt{B^*B}}{\sqrt{B^*B}} \\ -\frac{\sin \sqrt{B^*B}}{\sqrt{B^*B}} B^* & \cos \sqrt{B^*B} \end{pmatrix}.$$

Remark 6.1. Since

$$\begin{aligned}
(-BB^*)^{n+1} &= (-1)^{n+1} B(B^*B)^n B^*, \\
\cos \sqrt{BB^*} &= I_{k_1} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-BB^*)^{n+1} \\
&= I_{k_1} + B \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^{n+1} (B^*B)^n B^* \\
&= I_{k_1} + B(\sqrt{B^*B})^{-2} \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^{n+1} (\sqrt{B^*B})^{2n+2} B^* \\
&= I_{k_1} + B(\sqrt{B^*B})^{-2} (\cos \sqrt{B^*B} - I_{k_2}) B^*.
\end{aligned}$$

Remark 6.2. Since

$$\begin{aligned}
(-B^*B)^{n+1} &= (-1)^{n+1} B^*(BB^*)^n B, \\
\cos \sqrt{B^*B} &= I_{k_2} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-B^*B)^{n+1} \\
&= I_{k_2} + B^* \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^{n+1} (BB^*)^n B \\
&= I_{k_2} + B^*(\sqrt{BB^*})^{-2} \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^{n+1} (\sqrt{BB^*})^{2n+2} B \\
&= I_{k_2} + B^*(\sqrt{BB^*})^{-2} (\cos \sqrt{BB^*} - I_{k_1}) B.
\end{aligned}$$

Remark 6.3. $\cos \sqrt{B^*B}$, $\sqrt{B^*B}^{-1} \sin \sqrt{B^*B}$ and $(\sqrt{B^*B})^{-2} (\cos \sqrt{B^*B} - I_{k_2})$ are entire functions of B^*B .

Remark 6.4.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} K^{2n} K &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & (-BB^*)^n B \\ (-B^* B)^n (-B^*) & O_2 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & (-BB^*)^n B \\ (-B^*) (-B^* B)^n & O_2 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \begin{pmatrix} O_1 & (-1)^n \sqrt{BB^*}^{2n} B \\ (-B^*) (-1)^n \sqrt{BB^*}^{2n} & O_2 \end{pmatrix} \\
&= \begin{pmatrix} O_1 & \sqrt{B^* B}^{-1} \sin \sqrt{B^* B} \\ (-B^*) \sqrt{B^* B}^{-1} \sin \sqrt{B^* B} & O_2 \end{pmatrix}.
\end{aligned}$$

Let $X = B \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}}$ and $Y = \cos \sqrt{B^* B}$. Then

$$X^* X = \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} B^* B \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} = \sin^2 \sqrt{B^* B},$$

and

$$X^* X + Y^2 = \sin^2 \sqrt{B^* B} + \cos^2 \sqrt{B^* B} = I, \quad Y = (I - X^* X)^{1/2}.$$

In the same way, we have

$$\begin{aligned}
X X^* &= B \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} \frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}} B^* \\
&= B \frac{\sin^2 \sqrt{B^* B}}{B^* B} B^* = \frac{\sin^2 \sqrt{BB^*}}{BB^*} B B^* = \sin^2 \sqrt{BB^*},
\end{aligned}$$

where we used the fact that for an entire function $f(x) = \sum_{n=0}^{\infty} a_n x^n$,

$$B f(B^* B) B^* = \sum_{n=0}^{\infty} a_n B (B^* B)^n B^* = \sum_{n=0}^{\infty} a_n (B B^*)^n B B^* = f(B B^*) B B^*,$$

and $\frac{\sin \sqrt{B^* B}}{\sqrt{B^* B}}$ is an entire function of $B^* B$. This shows

$$\cos \sqrt{BB^*} = (I - X X^*)^{1/2}.$$

Since $K(B) \in \mathfrak{u}(n)$, $\exp K(B) \in U(n)$ and

$$(6.1) \quad \exp K(B) = \begin{pmatrix} (I_{k_1} - X X^*)^{1/2} & X \\ -X^* & (I_{k_2} - X^* X)^{1/2} \end{pmatrix} = W(X) \in U(n).$$

Without knowing such background, we can show directly the unitarity of the matrix (6.1).

Proposition 6.5. For $X \in M(k_1, k_2)$, $X^* X \leq I_{k_2} \Leftrightarrow X X^* \leq I_{k_1}$.

Proof. Here is the elementary proof.

$$\begin{aligned}
X^* X \leq I_{k_2} &\Leftrightarrow \forall e \in \mathbb{C}^{k_2} (\|e\| = 1 \Rightarrow (e, X^* X e)_2 \leq (e, I_{k_2} e)_2 = 1) \\
&\Leftrightarrow \forall e \in \mathbb{C}^{k_2} (\|e\| = 1 \Rightarrow (X e, X e)_1 \leq 1) \Leftrightarrow \forall e \in \mathbb{C}^{k_2} (\|e\| = 1 \Rightarrow \|X e\|_1 \leq 1) \\
&\Leftrightarrow \forall e \in \mathbb{C}^{k_2}, \forall d \in \mathbb{C}^{k_1} (\|e\| = \|d\| = 1 \Rightarrow (d, X e)_1 \leq 1) \\
&\Leftrightarrow \forall e \in \mathbb{C}^{k_2}, \forall d \in \mathbb{C}^{k_1} (\|e\| = \|d\| = 1 \Rightarrow (X^* d, e)_2 \leq 1) \\
&\Leftrightarrow \forall d \in \mathbb{C}^{k_1} (\|d\| = 1 \Rightarrow \|X^* d\|_2 \leq 1) \Leftrightarrow \forall d \in \mathbb{C}^{k_1} (\|d\| = 1 \Rightarrow (X^* d, X^* d)_2 \leq 1) \\
&\Leftrightarrow \forall d \in \mathbb{C}^{k_1} (\|d\| = 1 \Rightarrow (d, X X^* d)_1 \leq 1) \Leftrightarrow X X^* \leq I_{k_1}.
\end{aligned}$$

□

Proposition 6.6. *Let $X \in \bar{B}(k_1, k_2) = \{X \in M(k_1, k_2); X^*X \leq I_{k_2}\}$. Then $(I_{k_2} - X^*X)^{1/2}$ and $(I_{k_1} - XX^*)^{1/2}$ are well defined, and $W(X)$ of (6.1) which appeared in Proposition 2.4 is unitary.*

Proof.

$$\begin{aligned} & \begin{pmatrix} (I_{k_1} - XX^*)^{1/2} & -X \\ X^* & (I_{k_2} - X^*X)^{1/2} \end{pmatrix} \begin{pmatrix} (I_{k_1} - XX^*)^{1/2} & X \\ -X^* & (I_{k_2} - X^*X)^{1/2} \end{pmatrix} \\ &= \begin{pmatrix} I_{k_1} - XX^* + XX^* & (I_{k_1} - XX^*)^{1/2}X - X(I_{k_2} - X^*X)^{1/2} \\ X^*(I_{k_1} - XX^*)^{1/2} - (I_{k_2} - X^*X)^{1/2}X^* & X^*X + I_{k_2} - X^*X \end{pmatrix}. \end{aligned}$$

Since $(1+x)^\alpha$ for $\alpha = 1/2$ is expanded as $1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} x^n$ for $-1 < x < 1$,

$$\begin{aligned} (I_{k_1} - XX^*)^{1/2}X &= (I_{k_1} + \sum_{n=1}^{\infty} \binom{\alpha}{n} (-XX^*)^n)X \\ &= X + \sum_{n=1}^{\infty} \binom{\alpha}{n} X(-X^*X)^n = X(I_{k_1} - X^*X)^{1/2} \end{aligned}$$

and

$$\begin{aligned} X^*(I_{k_1} - XX^*)^{1/2} &= X^*(I_{k_2} + \sum_{n=1}^{\infty} \binom{\alpha}{n} (-XX^*)^n) \\ &= X^* + \sum_{n=1}^{\infty} \binom{\alpha}{n} (-X^*X)^n X^* = (I_{k_2} - X^*X)^{1/2} X^* \end{aligned}$$

hold for $X \in B(k_1, k_2) = \{X \in M(k_1, k_2); X^*X < I_{k_2}\}$. In order to show the above two formulae for $X \in \bar{B}(k_1, k_2)$, we employ a limiting process $B(k_1, k_2) \ni X_n \rightarrow X \in \bar{B}(k_1, k_2)$ as $n \rightarrow \infty$ with respect to a norm $\|X\|^2 = \text{Tr } X^*X = \sum_{i=1}^{k_1} \sum_{j=1}^{k_1} |x_{ij}|^2$. Thus we have $W(X)^*W(X) = I_n$ for all $X \in \bar{B}(k_1, k_2)$. \square

Remark 6.7. The following relation is useful. Here $\alpha = 1/2$.

$$\begin{aligned} (I_{k_1} - XX^*)^{1/2} &= I_{k_1} - X \sum_{n=1}^{\infty} \binom{\alpha}{n} (-X^*X)^{n-1} X^* \\ &= I_{k_1} + X(X^*X)^{-1/2} \sum_{n=1}^{\infty} \binom{\alpha}{n} (-X^*X)^n (X^*X)^{-1/2} X^* \\ &= I_{k_1} + X(X^*X)^{-1/2} [(1 - X^*X)^{1/2} - 1] (X^*X)^{-1/2} X^*. \end{aligned}$$

Remark 6.8. Since a Lie algebra describes only the local properties of its Lie group, the mapping $B \rightarrow \pi(\exp K(B))$ gives a local homeomorphism, that is, there is a neighborhood V of 0 in $M(k_1, k_2)$ and U of e in $U(n)$ such that $V \ni B \rightarrow \pi(\exp K(B)) \in \pi(U)$ is homeomorphic (see [13]). But for $W(X)$, Proposition 2.2 says that the mapping $\kappa_\sigma : B(k_1, k_2) \ni X \rightarrow \pi(W(X) \in \Omega_\sigma \subset \pi(U(n)))$ is bijective and Proposition 4.14 says the mapping $\bar{\kappa} : \bar{B}(k_1, k_2) \ni B \rightarrow \pi(W(X) \in \pi(U(n)))$ is surjective.

7. EXAMPLES

Here we give two examples of the parametrization of degenerate density matrices with diagonal matrices of eigenvalues of the forms:

1) $D_4(\boldsymbol{\lambda}) = \text{diag}_4(\lambda_1 I_3, \lambda_2 I_1)$,

2) $D_4(\boldsymbol{\lambda}) = \text{diag}_4(\lambda_1 I_2, \lambda_2 I_2)$.

For the first case, the density matrices are parametrized by

$$\Lambda_2^\neq \times G(1, \mathbb{C}^4) = \Lambda_2^\neq \times CP^3$$

(see (2.17)).

Since $\Omega_4 \subset CP^3$ is an open dense subset of CP^3 and Ω_4 is parametrized by $B(3) = \{z \in \mathbb{C}^3; |z| < 1\}$, almost all density matrices are parametrized by $\Lambda \times B(3)$.

Concretely, we have the following parametrization:

$$\begin{aligned} \Lambda_2^\neq \times B(3) \ni ((\lambda_1, \lambda_2), x) \rightarrow \rho = \\ \begin{pmatrix} (I_3 - xx^*)^{1/2} & x \\ -x^* & (1 - x^*x)^{1/2} \end{pmatrix} \begin{pmatrix} \lambda_1 I_3 & 0 \\ 0 & \lambda_2 I_1 \end{pmatrix} \begin{pmatrix} (I_3 - xx^*)^{1/2} & x \\ -x^* & (1 - x^*x)^{1/2} \end{pmatrix}^* . \end{aligned}$$

$(I_3 - xx^*)^{1/2}$ can be calculated as:

$$(I_3 - xx^*)^{1/2} = I_3 + (x^*x)^{-1}[(1 - x^*x)^{1/2} - 1]xx^*$$

(see Remark 6.7).

For the second case, the density matrices are parametrized by

$$\Lambda_2^\neq \times G(2, \mathbb{C}^4).$$

Since $\Omega_e \subset G(2, \mathbb{C}^4)$ is an open dense subset of $G(2, \mathbb{C}^4)$ and Ω_e is parametrized by $B(2, 2) = \{X \in M(2, 2); X^*X < I_2\}$, almost all density matrices are parametrized by $\Lambda \times B(2, 2)$.

Concretely, we have the following parametrization:

$$\begin{aligned} \Lambda_2^\neq \times B(2, 2) \ni ((\lambda_1, \lambda_2), x) \rightarrow \rho = \\ \begin{pmatrix} (I_2 - xx^*)^{1/2} & x \\ -x^* & (I_2 - x^*x)^{1/2} \end{pmatrix} \begin{pmatrix} \lambda_1 I_2 & 0 \\ 0 & \lambda_2 I_2 \end{pmatrix} \begin{pmatrix} (I_2 - xx^*)^{1/2} & x \\ -x^* & (I_2 - x^*x)^{1/2} \end{pmatrix}^* . \end{aligned}$$

But unfortunately, $(I_2 - xx^*)^{1/2}$ and $(I_2 - x^*x)^{1/2}$ are not easy to calculate. So, we employ Corollary 5.4 which states that there is a bijection $\phi_e : \Omega_e \rightarrow \Omega_{2,4} \times \Omega_{2,3} \subset CP^3 \times CP^2$, and $\psi \circ \phi_e$ is a local section of $U(4)$ on Ω_e , where

$$\psi(z_2, z_1) = \iota_4(z_2) \begin{pmatrix} \iota_3(z_1) & 0 \\ 0 & I_1 \end{pmatrix},$$

and ι_j is the section defined by (3.8). Concretely, we have the following parametrization:

$$\begin{aligned} \Lambda_2^\neq \times B(2)^2 \ni (\lambda_1, \lambda_2, x_2, x_1) \rightarrow \rho = U(x_2, x_1) \text{diag}_4(\lambda_1 I_2, \lambda_2 I_2) U(x_2, x_1)^*, \\ U(x_2, x_1) = \begin{pmatrix} (I_2 - x_2 x_2^*)^{1/2} & 0 & x_2 \\ 0 & 1 & 0 \\ -x_2^* & 0 & (1 - x_2^* x_2)^{1/2} \end{pmatrix} \begin{pmatrix} (I_2 - x_1 x_1^*)^{1/2} & x_1 & 0 \\ -x_1^* & (1 - x_1^* x_1)^{1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

8. CONCLUSION

The problem of parametrizing degenerate density matrices required to develop a new approach using techniques from the theory of homogeneous spaces as outlined in sections 1 - 5. This approach is not based on the use of Lie algebra methods. Actually our approach helps to detect some shortcomings of the Lie algebra approach as used in the given references, i.e., the exponential map from Lie algebra to Lie group is not one to one and onto, and to correct these, also in the case of non-degenerate density matrices. These shortcomings are due to the non-injectivity of the given map at the boundary of the respective parameter domain.

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